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# Soliton Solution for the Korteweg-de Vries Equation

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## Abstract

Korteweg de Vries (KdV) model is considered quintessential in modeling the surface gravity water waves in shallow water. In this project, we are interested in starting from the Elliptic Jacobian Functions, [4], and performing a complete analysis of these functions to discover that one can recover the soliton in the particular case  $m \rightarrow 1$ , where  $m$  is a parameter between 0 and 1 in the definition of the Elliptic Jacobian Functions. This analysis will provide us with an understanding of cnoidal periodic waves and how, through them, we can derive the soliton solution. Finally, this project grants readers a deeper understanding of the origin of solitons and their applications in water wave theory.

**Keywords:** KdV, water waves, cnoidal waves, periodic waves, nonlinear waves, solitons.

## 1 Introduction to Solitons and Water Wave Theory

In September 1844, in York, England, John Scott Russel reported to the British Association of the Advancement of Science, the discovery of a novel class of wave which the writer has called “the great wave of translation”, [10]. He described it as a wave consisting of a single elevation, which, if properly started, may travel for a considerable distance along a uniform channel, with little or no change. In the fluid dynamics literature, Russell’s wave is referred to as Russell’s solitary wave.

The solitary wave has the following characteristics

- It is a long, shallow wave of permanent form, i.e., its amplitude ( $a$ ) is small compared to its wavelength ( $\lambda$ ):  $\frac{a}{\lambda} \ll 1$ .
- The speed of propagation,  $c$ , is given by

$$c^2 = g(h + a) \quad (1)$$

where  $g$  is the gravitational acceleration, and  $h$  is the constant depth of a long and narrow channel. The equation (1) tells us that higher solitary waves travel faster.

Figure 1 shows a solitary wave subject to gravitational acceleration  $g$  in a channel of uniform depth  $h$ . The speed of the solitary wave is given by (1).

At the time, Russell’s observations came in conflict with Airy’s wave theory developed in 1841, which predicted that a small amplitude wave cannot propagate without change of profile when it propagates in constant finite depth. The article “The Origins of Water Wave Theory”, [3], gives the insight on how Airy viewed Russel’s report from 1844.

Boussinesq, in 1871, and Lord Rayleigh, in 1876, explained Russel's findings. The article "On the solitary wave", [6], gives the insight on the supportive theory Boussinesq and Lord Rayleigh developed separately to explain Russell's solitary wave. Both Boussinesq and Lord Rayleigh, developed the supportive theory for Russell's findings using the equations of motion for an ideal fluid, i.e., incompressible and inviscid. As well, they assumed that Russell's solitary wave is a class of wave with a wavelength ( $\lambda_0$ ), much greater than the depth of water ( $h$ ), i.e.,

$$\delta^2 := \left(\frac{h}{\lambda_0}\right)^2 = O(\varepsilon), \quad \varepsilon \ll 1 \quad (2)$$

where  $\delta^2$  is called the square of the frequency dispersion parameter.

Boussinesq, [2], showed that appropriate allowance for the vertical acceleration, which is responsible for dispersion and is neglected in Airy's wave theory, and appropriate allowance for the finite amplitude, leads to the following solution for the profile  $z = \eta(x, t)$  of the free surface (i.e., undisturbed level) elevation

$$\eta(x, t) = a \operatorname{sech}^2(\beta(x - ct)), \quad \beta^2 = \frac{1}{\lambda_0^2} = \frac{3a}{4h^2(h + a)} \quad (3)$$

for any  $a > 0$ .

He discovered as well that the profile (3) is correct only if (2) is satisfied.

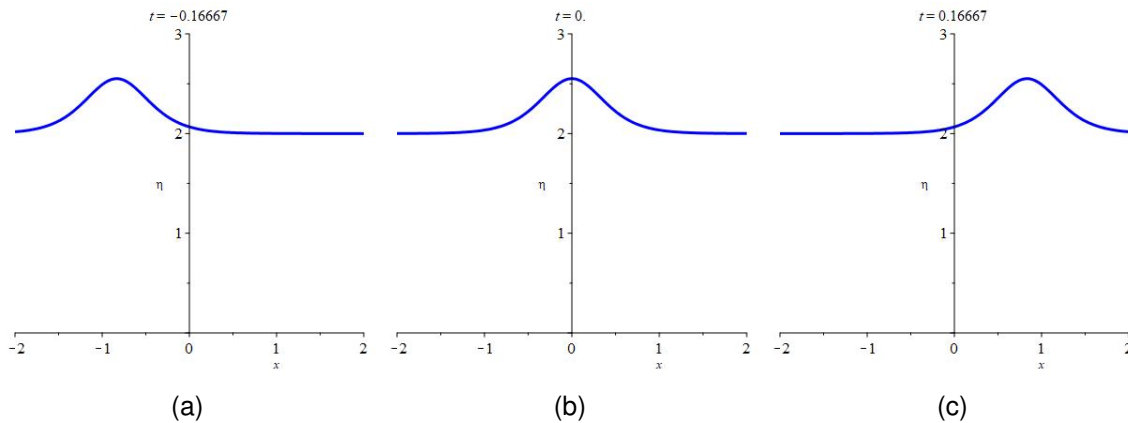


Figure 1: The time evolution of Russell's solitary wave as explained by Boussinesq for  $c = 5$ ,  $g = 9.8$ , and  $h = 2$ .

**Note:** The profile of the solitary wave is shown from the free surface elevation, i.e., Boussinesq's solution was raised by the depth,  $h$ , of the channel.

Lord Rayleigh, [9], treated the problem as one of steady motion (i.e., time independent), and he derived the following ordinary differential equation (ODE)

$$\left(\frac{dy}{dx}\right)^2 = 3\frac{(y - h)^2}{h^2} \left(1 - \frac{gy}{c^2}\right) \quad (4)$$

The ODE (4) governs long one-dimensional, small amplitude, surface gravity waves in a channel of water of uniform depth  $h$ , where  $c$  is the uniform velocity in the parts of the fluid at a distance from

the wave, whether in front or behind it. From the ODE (4) we can immediately see that the only elevations in the solution curves of the equation, when the tangents to the curves are horizontal, happen at the free surface, i.e.,  $y = h$ , or when  $y = \frac{c^2}{g}$ . Since  $1 - \frac{gy}{c^2}$  is non-negative, we obtain that the maximum elevation of any solution curve of the ODE (4) where the tangent is horizontal, is  $y_{max} = \frac{c^2}{g}$ , and consequently we have no depression (i.e., underneath the free surface) in the profile of the wave. Therefore, the wave is necessarily of one elevation only. Denoting by  $a$  the maximum height above the free surface, we get

$$c^2 = gy_{max} = g(h + a) \tag{5}$$

which represents (1).

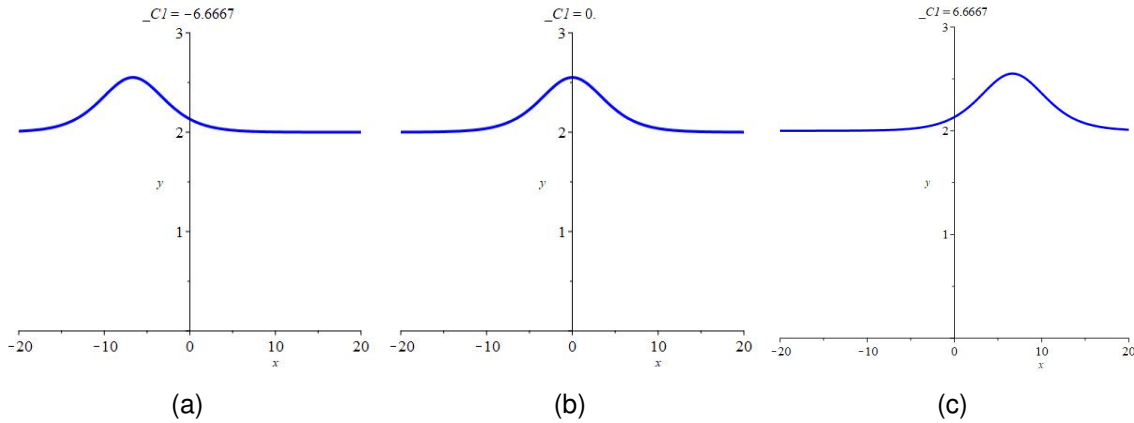


Figure 2: Russell’s solitary wave as explained by Lord Rayleigh for  $c = 5$ ,  $g = 9.8$ , and  $h = 2$ .

Lord Rayleigh’s solution takes in account the depth,  $h$ , of the channel. The value of the constant of integration obtained by solving the ODE (4) shifts horizontally the solitary wave profile as it would propagate with respect to time.

**Concluding remark on the works of Boussinesq and Lord Rayleigh regarding Russell’s solitary wave**

Distinct from Lord Rayleigh’s article, Boussinesq introduces a time variable, essential for the description of a dynamic phenomenon. Hence, Boussinesq’s work is more comprehensive in describing Russell’s solitary wave, as he used a complex evolution partial differential equation (PDE) to derive the profile of Russell’s solitary wave (3). As well, Boussinesq’s work reveals something unique about the nature of Russell’s solitary wave. Using the Ursell number,  $U$ , which is a dimensionless parameter used in fluid dynamics to measure the nonlinearity of long surface gravity waves, we obtain

$$U = \frac{a\lambda_0^2}{h^3} = \frac{\frac{a}{h}}{\left(\frac{h}{\lambda_0}\right)^2} = \frac{\frac{a}{h}}{\delta^2} = \frac{O(\varepsilon)}{O(\varepsilon)} = O(1) \tag{6}$$

The relationship (6) tells us that the solitary waves have the essential quality of balance between nonlinearity (measured by  $a/h$ ) and dispersion (measured by  $\delta^2 = (h/\lambda_0)^2$ ).

## 2 Korteweg-de Vries Equation

In 1895, Korteweg and de Vries, who apparently did not know of the work of Boussinesq and Lord Rayleigh, derived a nonlinear evolution partial differential equation, known as the Korteweg-de Vries (KdV) equation, governing long one dimensional, small-amplitude, surface gravity waves propagating in shallow water of uniform depth.

The KdV equation in the dimensional form can be represented through the following PDE:

$$\eta_t + \frac{3}{2}c_0\left(\frac{1}{h}\eta\eta_x + \frac{2}{3}\alpha\eta_x + \frac{1}{3}\sigma\eta_{xxx}\right) = 0 \quad (7)$$

where  $\sigma = \frac{1}{3}h^2 - \frac{T}{\rho g}$ , and  $c_0 = \sqrt{gh}$ .

Explanations of the symbols in the KdV model (7)

- $\eta$  represents the surface elevation of the wave about the equilibrium level  $h$ .
- $\alpha$  represents the small arbitrary constant related to the uniform motion of the liquid
- $g$  represents the gravitational constant
- $\sigma$  is the parameter depending on the surface tension  $T$  of the liquid of constant density  $\rho$
- $c_0$  represents the wave velocity

**Remark**  $c_0$  represents the wave velocity only in the first order approximation, where  $a/h$  may be neglected. In the case when  $a/h$  cannot be neglected, it is impossible to have a wave in still water with velocity  $\sqrt{gh}$ , and at the same time propagating without change of form. Hence, in order to account for this discrepancy, a more accurate approximation of the wave velocity is used in nonlinear water wave theory.

In the first section we mentioned that Boussinesq, in his comprehensive work, used a complex evolution PDE to derive the profile of Russell's solitary wave (3).

The Boussinesq equation is

$$\eta_{tt} = c_0^2(\eta_{xx} + \frac{3}{2h}(\eta^2)_{xx} + \frac{1}{3}h^2\eta_{xxxx}) \quad (8)$$

where  $c_0 = \sqrt{gh}$

Boussinesq equation (8) describes one-dimensional weakly nonlinear dispersive water waves propagating in both directions in water of uniform depth  $h$ . A derivation of the KdV equation out of the Boussinesq equation was presented in [11], and is based on the Riemann invariants method applied by Zabusky and Kruskal, [12].

### 2.1 Periodic Solutions discovered by Korteweg and de Vries: The Cnoidal Waves

Before we start, let us note the Jacobian elliptic functions  $cn$  and  $sn$ , defined as follows, [8]

$$sn(v|m) = \sin \phi \quad \text{and} \quad cn(v|m) = \cos \phi \quad (9)$$

where  $v$  and  $\phi$  are related by the integral

$$v = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \quad m \in [0, 1]$$

Russel's solitary wave is recovered from the cnoidal waves in the special case  $m \rightarrow 1$ . Both the periodic and the solitary waves described by the KdV model (7) are found as solutions of constant shape moving with constant velocity ( $v$ ). Thus, we can describe them as a special type of travelling waves as follows:

$$\eta(x, t) = hu(x - vt) \quad (10)$$

We obtain partial derivatives of the function  $\eta(x, t)$  as follows

$$\begin{aligned} \eta_t &= h(-v)u' \\ \eta_x &= hu', \quad \eta_{xx} = hu'', \quad \eta_{xxx} = hu''' \end{aligned} \quad (11)$$

**Remark** Note that in (11),  $u'$  stands for  $\frac{du}{d(x-vt)}$ , such that we have, for instance,  $\frac{du}{dt} = (-v)u'$ . Then the KdV model (7) becomes

$$\begin{aligned} hu'(-v) + \frac{3}{2}c_0\left(\frac{1}{h}hu \cdot hu' + \frac{2}{3}\alpha hu' + \frac{1}{3}\sigma hu'''\right) &= 0 \implies \\ -hvu' + \frac{3}{2}c_0h\left(uu' + \frac{2}{3}\alpha u' + \frac{1}{3}\sigma u'''\right) &= 0 \implies \\ -\frac{2}{3}\frac{v}{c_0}u' + uu' + \frac{2}{3}\alpha u' + \frac{1}{3}\sigma u''' &= 0 \end{aligned} \quad (12)$$

Let us recall from the equation (7),  $\sigma = \frac{1}{3}h^2 - \frac{T}{\rho g}$ . Neglecting the tension  $T$ , we get  $\sigma = \frac{1}{3}h^2$ . Then the equation (12) becomes

$$\frac{h^2}{9}u''' + \frac{2}{3}\left(\alpha - \frac{v}{c_0}\right)u' + uu' = 0 \quad (13)$$

Integrating equation (13) once, we obtain

$$\frac{h^2}{9}u'' + \frac{2}{3}\left(\alpha - \frac{v}{c_0}\right)u + \frac{u^2}{2} + C_1 = 0 \quad (14)$$

where

$$\int uu' = \int \left(\frac{u^2}{2}\right)' = \frac{u^2}{2} + C_1.$$

Multiplying equation (14) by  $u'$ , we obtain the following

$$\frac{h^2}{9}u''u' + \frac{2}{3}\left(\alpha - \frac{v}{c_0}\right)uu' + \frac{u^2u'}{2} + C_1u' = 0 \quad (15)$$

Integrating equation (15) once more, we obtain

$$\begin{aligned}
\frac{h^2}{9} \frac{(u')^2}{2} + \frac{2}{3} \left( \alpha - \frac{v}{c_0} \right) \frac{u^2}{2} + \frac{1}{2} \frac{u^3}{3} + C_1 u + C_2 &= 0 \implies \\
\frac{h^2}{3} (u')^2 + 2 \left( \alpha - \frac{v}{c_0} \right) u^2 + u^3 + 6C_1 u + C_2 &= 0 \implies \\
\frac{h^2}{3} (u')^2 = -u^3 - 2 \left( \alpha - \frac{v}{c_0} \right) u^2 - 6C_1 u - C_2 &= 0 \implies \\
\frac{h^2}{3} (u')^2 = -u^3 + 2 \left( \frac{v}{c_0} - \alpha \right) u^2 + 6C_1 u + C_2 &
\end{aligned} \tag{16}$$

where

$$\int u'' u' = \int \left( \frac{(u')^2}{2} \right)' = \frac{(u')^2}{2} + C_2.$$

Let us notice that the right hand side of the equation (16) is a polynomial of degree 3, and let us denote it by  $P(u)$ . Then the equation (16) becomes

$$\frac{h^2}{3} (u')^2 = P(u) \tag{17}$$

Since we look for real bounded solutions only, we are interested in the area where  $P(u) \geq 0$ . Looking at the degree of polynomial  $P(u)$ , we notice that  $P(u)$  will either have one real root or three real roots. In this work, we will be interested in the case when  $P(u)$  will have three distinct real roots,  $u_1$ ,  $u_2$ , and  $u_3$ . Without loss of generality let us assume  $u_1 < u_2 < u_3$ . Then we will have

$$P(u) = -(u - u_1)(u - u_2)(u - u_3).$$

Then the ODE (17) becomes

$$\begin{aligned}
\frac{h^2}{3} (u')^2 = -(u - u_1)(u - u_2)(u - u_3) &\implies \\
\frac{h^2}{3} (u')^2 = (u_1 - u)(u - u_2)(u - u_3) &\implies \\
\sqrt{\frac{1}{3}} h u' = \pm \sqrt{(u_1 - u)(u - u_2)(u - u_3)} &
\end{aligned} \tag{18}$$

Rewriting  $u'$  as  $u' = \frac{du}{d(x-vt)}$ , from (18) we obtain

$$\sqrt{\frac{1}{3}} h \frac{du}{\sqrt{(u_1 - u)(u - u_2)(u - u_3)}} = \pm d(x - vt) \tag{19}$$

Integrating (19), we obtain

$$\begin{aligned}
\pm(x - vt) &= \int_{u_3}^u \sqrt{\frac{1}{3}} h \frac{dw}{\sqrt{(u_1 - w)(w - u_2)(w - u_3)}} \pm u_3 \\
x - vt &= u_3 \pm \sqrt{\frac{1}{3}} h \int_{u_3}^u \frac{dw}{\sqrt{(u_1 - w)(w - u_2)(w - u_3)}}
\end{aligned} \tag{20}$$

Performing the substitution,  $\chi = x - vt$ , the equation (20) becomes

$$\chi = u_3 \pm \sqrt{\frac{h}{3}} \int_{u_3}^u \frac{dw}{\sqrt{(u_1 - w)(w - u_2)(w - u_3)}} \quad (21)$$

Let us make the following substitution

$$w = u_3 + (u_2 - u_3) \sin^2 \theta \quad (22)$$

Then we have

$$dw = (u_2 - u_3) 2 \sin \theta \cos \theta d\theta \quad (23)$$

The new limits of integration of the integral in (21), in terms of the substitution (22), are

$$u = u_3 + (u_2 - u_3) \sin^2 \theta \implies \theta = \phi$$

and

$$u_3 = u_3 + (u_2 - u_3) \sin^2 \theta \implies \sin^2 \theta = 0 \implies \theta = 0$$

Then we have,

$$\begin{aligned} & \int_{u_3}^u \frac{dw}{\sqrt{(u_1 - w)(w - u_2)(w - u_3)}} = \\ & \int_0^\phi \frac{(u_2 - u_3) 2 \sin \theta \cos \theta d\theta}{\sqrt{(u_1 - u_3 + (u_3 - u_2) \sin^2 \theta)(u_3 - u_2 + (u_2 - u_3) \sin^2 \theta)(u_2 - u_3) \sin^2 \theta}} = \\ & \int_0^\phi \frac{-2d\theta}{\sqrt{(u_3 - u_1) \left(1 - \frac{u_3 - u_2}{u_3 - u_1} \sin^2 \theta\right)}} = \\ & \int_0^\phi \frac{-2d\theta}{\sqrt{(u_3 - u_1) (1 - m \sin^2 \theta)}} \end{aligned} \quad (24)$$

Where we denote  $m = \frac{u_3 - u_2}{u_3 - u_1}$ ,  $m \in (0, 1)$ .

Hence we have,

$$\begin{aligned} \chi &= u_3 \pm \frac{h}{\sqrt{3}} \int_0^\phi \frac{-2d\theta}{\sqrt{(u_3 - u_1) (1 - m \sin^2 \theta)}} \implies \\ & \mp (\chi - u_3) \frac{\sqrt{3(u_3 - u_1)}}{2h} = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \end{aligned} \quad (25)$$

The equation (25) is nothing else but a  $cn$  even function for  $v = (\chi - u_3) \frac{\sqrt{3(u_3 - u_1)}}{2h}$ , as defined in (9). Hence, we can write it as

$$cn \left[ (\chi - u_3) \frac{\sqrt{3(u_3 - u_1)}}{2h} \mid m \right] = \cos \theta \quad (26)$$

From the substitution that was preformed above we have

$$\begin{aligned}
 u &= u_3 + (u_2 - u_3) \sin^2 \phi \\
 &= u_2 + (u_3 - u_2) \cos^2 \phi \\
 &= u_2 + (u_3 - u_2) cn^2 \left[ (x - u_3) \frac{\sqrt{3(u_3 - u_1)}}{2h} \mid m \right]
 \end{aligned}
 \tag{27}$$

We have  $\eta(x, t) = hu(\chi) = hu(x - vt)$ , then the equation (27) becomes

$$\eta(x, t) = h \left[ u_2 + (u_3 - u_2) cn^2 \left[ \frac{\sqrt{3(u_3 - u_1)}}{2h} (x - vt - u_3) \mid m \right] \right]
 \tag{28}$$

The formula above is called the cnoidal-wave solution of the KdV model (7). In the limit case

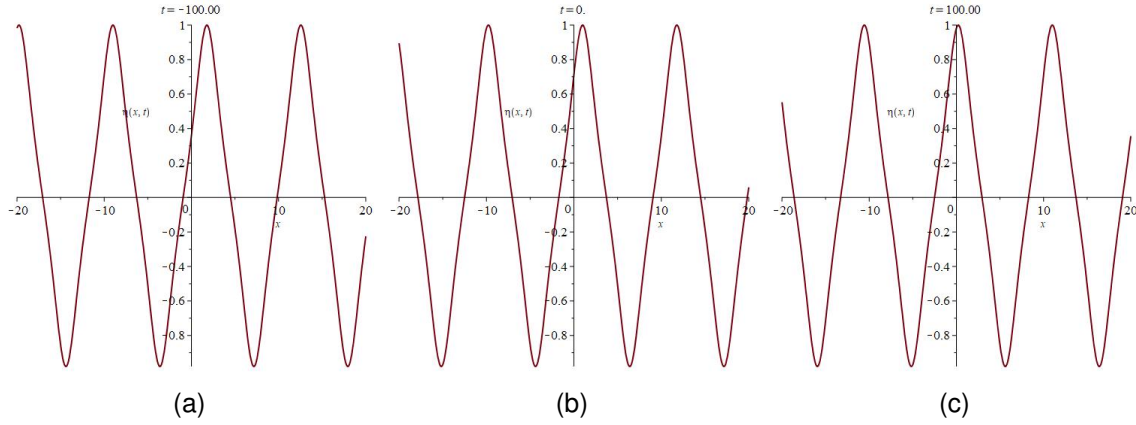


Figure 3: The time evolution of a periodic cnoidal wave for the KdV model (7), following the formula (28) for  $h = 1$ ,  $u_3 = 1$ ,  $u_2 = 0.1$ ,  $u_1 = 0.01$ , and  $v = 0.1$ .

$m \rightarrow 1$  (i.e.,  $u_2 \rightarrow u_1$ ), we have  $cn \rightarrow \text{sech}$ , and the Russell's solitary wave is recovered.

$$\eta(x, t) = h \left[ u_1 + (u_3 - u_1) \text{sech}^2 \left[ \frac{\sqrt{3(u_3 - u_1)}}{2h} (x - vt - u_3) \right] \right]
 \tag{29}$$



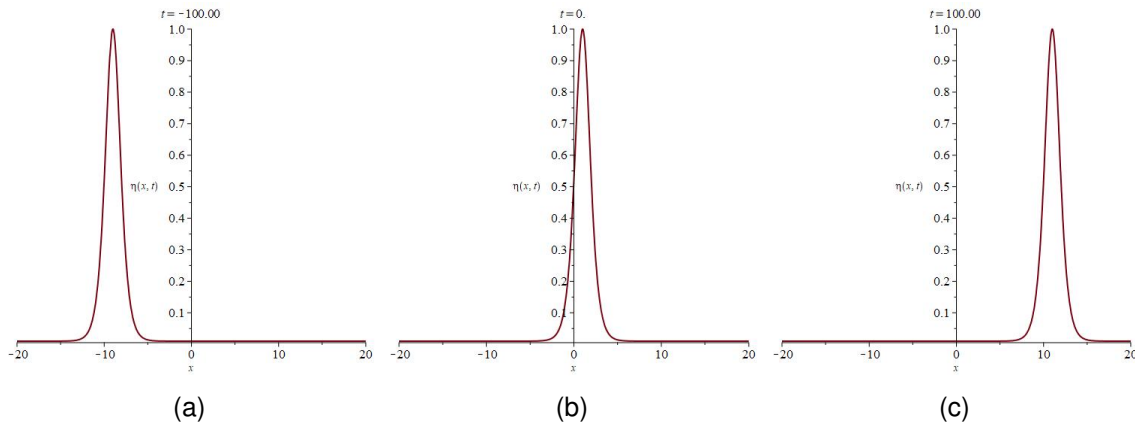


Figure 4: The time evolution of a solitary wave for the KdV model (1), following the formula (29) for  $h = 1$ ,  $u_3 = 1$ ,  $u_2 = u_1 = 0.01$ , and  $v = 0.1$ .

### 3 Summary and Discussions

In this paper we introduced the Korteweg-de Vries equation, which was derived as a model for long wave propagation in shallow water, and it is a great example for explaining the propagation of weakly dispersive and weakly nonlinear waves. In recent years, KdV equation has been used as a model for a variety of physical phenomena, ranging from plasma and solid state physics to biology, since it requires only a few general assumptions about the structure of nonlinearity, and dispersion to be made [7].

The first part of this paper mainly focuses on the core ideas regarding the nature of the solitary waves, starting with experiments of John Scott Russell in 1834, followed by theoretical investigations of Boussinesq in 1871 and Lord Rayleigh in 1876, and finally, Korteweg and de Vries in 1895. In the later part of this paper we derived the nonlinear and exact periodic wave solution of the Korteweg–de Vries equation, known as the cnoidal waves. These solutions are given in terms of the Jacobi elliptic function  $cn$ , and are used to describe surface gravity waves in shallow water. Finally, the solitary wave solution, i.e., the soliton, is recovered from the cnoidal wave in the limiting case  $m \rightarrow 1$ , when  $cn \rightarrow \text{sech}$ . The method for solving the KdV equation presented in this paper is heavily based on the original work of Korteweg and de Vries dating back to 1895. However, the modern theory of soliton was actually developed much later in 1967, when Gardner, Greene, Kruskal and Miura related KdV equation to the inverse scattering problem for a one-dimensional linear Schrödinger equation. It is worth noting that Mark Ablowitz, in his book *Nonlinear Dispersive Waves*, [1], gives a comprehensive explanation of the inverse scattering problem for the KdV equation.

### 4 Acknowledgments

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