Generating Functions Related to the Fibonacci Substitution

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Abstract

In this paper, two generating function representations of the Fibonacci Substitution Tiling are derived and proven to converge on the interval $-1 < x < 1$. A sequence of signs for the Fibonacci Substitution is established along with a conjecture that the interval of convergence has an infinite number of zeroes.

1 Introduction

In his 1202 book *Liber Abaci*, Fibonacci discusses the "Problem of the Rabbits" [4][9]. This rabbit problem seeks to determine the number of offspring that can be produced in one year, beginning with one pair of baby rabbits of opposite sex. As Grimaldi [9, Chapter 2] notes, the problem has the following initial conditions:

1. all baby rabbits mature to adulthood in 2 months, then reproduce at the start of each new month
2. every pair of baby rabbits consists of one male and one female
3. no rabbits die throughout the year

For instance, in March the initial pair of rabbits are adults and reproduce, giving one pair of baby rabbits. Then in April, the initial pair has another pair of baby rabbits and the baby rabbits from the month prior become adults, giving two pairs of adult rabbits and one pair of baby rabbits. Throughout this process, Fibonacci found that the number of baby, adult and total number of rabbits each followed the same sequence of numbers {0, 1, 1, 2, 3, 5, 8, 13, 21, . . .} [9]. He also demonstrated that each number in this sequence was the sum of its two previous numbers [4]. As a result, these numbers became known as the Fibonacci numbers and their sequence the Fibonacci sequence.

The use of the Fibonacci sequence is one way to solve the rabbits problem; however, the problem can also be found through a substitution rule. As has been established, adult rabbits reproduce, leaving an adult and a baby pair; baby pairs of rabbits grow up to become adult rabbits. Thus the substitution rule becomes,

$$\begin{aligned} \sigma : a &\to ab \\ b &\to a \end{aligned}$$

where a represents the adult rabbits and b represents the baby rabbits. Through repeated concatenations of this rule, a subword with the total amount of adult and baby rabbits is derived. We know from the example above that March has one pair of adult rabbits who give birth to one pair

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of baby rabbits. Thus, the subword for March is ab. In April, the initial adult pair has another set of babies and the previous month's babies grow up, making the subword aba. By using this substitution rule, the rabbits problem is being solved via a substitution tiling; specifically, the one-sided Fibonacci Substitution tiling [1][7].

The implications of this Fibonacci Substitution can be demonstrated through a real-life example of the rabbits problem. In 1859, an English farmer named Thomas Austin imported 12 breeding pairs of rabbits into Australia, letting them loose to "naturalize" to the environment [5]. Within 70 years, 70 percent of the Australian continent was overrun with rabbits [18]: in 1948 alone, 130 million rabbits were killed in an attempt to control the massive population size [5]. Despite modern control measures, there are still an estimated 200 million feral rabbits in Australia today, showing the infinite concatenation process found in the Fibonacci Substitution [14].

The focus of this paper is on the one-sided Fibonacci Substitution. We begin in section 2 by briefly reviewing the Fibonacci numbers, substitution tilings, the Fibonacci Substitution and generating functions. In section 3 we use two generating functions to encode the Fibonacci Substitution and derive each of their recurrence relation formulas. Properties for these generating functions such as the interval of convergence are shown and proven. We complete the paper in section 4 by showing the existence of a sequence of signs recurrence relation for the generating functions, that the Fibonacci Substitution arises in this recurrence and by posing the conjecture that there are an infinite amount of zeroes in the interval $(-1, 1)$.

2 Preliminaries

In this section, we briefly review Fibonacci numbers, the one-sided Fibonacci Substitution and generating functions.

Fibonacci Numbers are the terms in the sequence $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$ where each term is the sum of the two previous terms, starting with $f_0 = 0$ and $f_1 = 1$ (see for example [10, 6]). This process of repeatedly adding the two previous terms gives the Fibonacci numbers as the following linear recurrence relation,

Definition 2.1 (**Fibonacci Numbers**). *Let $f_0 = 0$ and $f_1 = 1$. Then the* ***Fibonacci Numbers*** *are defined recursively as,*

$$f_{n+1} = f_n + f_{n-1}$$

Next we describe the notion of substitution tilings, focusing mainly on the one-sided Fibonacci Substitution. For a general introduction to other substitution tilings we recommend [1][8].
A substitution tiling is a rule that uses a finite number of tiles to construct an infinite tiling of the Euclidean Space $\mathbb{R}^d$ through repeated substitutions [7]. In the 'cut and project' method the tiles are typically represented by different shapes or lines. This geometric representation of tilings is useful in modelling physical quasi-crystalline materials as certain sequences are highly ordered and aperiodic, which is similar to the structures of these quasi-crystals [2][12][16][13]. Thus the point-sets provided from these tilings can provide interesting diffraction and symmetry properties on quasi-crystals [13].

In combinatorial representations, tiles are often represented by letters that are repeatedly concatenated to create finite subwords [13]. The Fibonacci Substitution is one instance of a substitution tiling that can be viewed this way. The Fibonacci Substitution is an aperiodic substitution tiling that has the letters 'a' and 'b' as its tiles [2]. Its inflation rule is as follows,

Definition 2.2 (The Fibonacci Substitution Inflation Rule)**.** *[1, Example 4.6] The* ***one-sided Fibonacci Substitution*** *abides by the following inflation rule:*

$$\sigma : a \to ab$$
$$b \to a$$

where a and b are the tiles in the substitution.

This inflation rule repeatedly concatenates the string of letters, providing a finite subword for each iteration. Similar to the Fibonacci Numbers (Definition 2.1), each iteration's string of letters is composed of the subwords from the previous two iterations, giving the Fibonacci Substitution the following recursive definition:

Definition 2.3 (The Recursive Fibonacci Substitution)**.** *[17] Let F_n represent the n^{th} iteration in the Fibonacci Substitution. Let $F_0 = b$ and $F_1 = a$. Then for $n \geqslant 2$ the Fibonacci Substitution is defined recursively by concatenation as,*

$$F_{n+1} = F_n F_{n-1}$$

By using Definition 2.3, the first 8 finite subwords of the Fibonacci Substitution are,

$$F_0 = b$$
$$F_1 = a$$
$$F_2 = ab$$
$$F_3 = aba$$
$$F_4 = abaab$$
$$F_5 = abaababa$$
$$F_6 = abaababaabaab$$
$$F_7 = abaababaabaababaababa$$

where the pink shows the newly appended section.

In the examples of F_n above, we see that there are never two 'b' tiles in a row. Below it will be shown that F_n always contains this property.

Proposition 2.4. *F_n never has two successive b tiles.*

Proof of Proposition 2.4. First observe that $F_0 = b$ and $F_1 = a$ do not have two successive 'b' tiles. For $n \geqslant 2$, assume that F_m does not have two consecutive 'b' tiles for every $1 \leqslant m \leqslant n$. Now, $F_{n+1} = F_n F_{n-1}$ and the appended F_{n-1} must begin with an 'a' value due to Definition 2.2 and since $n - 1 \geqslant 1$. Due to this occurrence, there can not be two successive 'b' tiles where F_n and F_{n-1} join together regardless if F_n ends in either an 'a' or a 'b'. Within F_n and F_{n-1} no two consecutive 'b' tiles can occur by hypothesis. Therefore, no two consecutive 'b' tiles can occur in F_{n+1}. By induction the result follows. ■

By using Definition 2.3, it is easy to see that F_n never ends in aa. Then, similarly to Proposition 2.4 we can prove that,

Proposition 2.5. *There can not be three successive 'a' values in any F_n.*

By using Definition 2.3, the subsequent well known result immediately follows,

Proposition 2.6. *Each $|F_n|$ has f_n 'a' values, f_{n-1} 'b' values and f_{n+1} total terms.*

As the iterations occur, the subwords of F_n become quite large and hard to work with; however, there are different ways to represent the iterations so that they are not large strings of letters. One such way is with generating functions, which we will review next.

A generating function converts an infinite sequence into a function that can be manipulated in ways that could not be done with the original sequence, allowing new information to be gained on the sequence [11]. The functions that these sequences are transformed into are real-valued functions, which are dealt with more commonly than abstract sequences and thus have many mathematical tools at their use for their analysis. In discrete mathematics, this conversion from abstract into real through generating functions allows use of these real-valued math tools on abstract sequences. There are many types of generating functions, but we will focus on Ordinary Generating Functions.

Definition 2.7 (**Ordinary Generating Functions**). *[3, Page 269]*
Suppose we have a sequence $\{a_0, a_1, a_2, a_3, \ldots\}$. The ***ordinary generating function (OGF)*** *associated with this sequence is the function whose value at x is $\sum_{n=0}^{\infty} a_n x^n$. The sequence $\{a_0, a_1, a_2, a_3, \ldots\}$ is called the coefficients of the generating function.*

While the generating function is called a function, it does not output the n^{th} term in an infinite sequence but instead a function whose power series displays the terms of the sequence as shown below [11][19]:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

An expanded generating function is one representation of a sequence, which in rare cases can be used to find a closed form representation of the sequence that has a finite number of terms. For instance, the sequence $\{1, 1, 1, 1, \ldots\}$ has the form,

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

where $\frac{1}{1-x}$ is the closed form representation of the sequence. The Fibonacci Numbers also have a closed form equation. To find this form, we first look at the expanded generating function,

$$\sum_{n=0}^{\infty} f_n x^n = 0x^0 + 1x^1 + 1x^2 + 2x^3 + \ldots = \frac{1}{1-x-x^2}$$

These functions can then be used to deduce the closed form representation **Binet's Formula**:

$$f_n = \frac{1}{\sqrt{5}}[\varphi^n - (-\varphi^{-n})]$$

where φ is the golden ratio $\frac{1+\sqrt{5}}{2}$ and $\frac{1}{-\varphi}$ is the golden root conjugate $\frac{1-\sqrt{5}}{2}$ [1][15][19].

In this paper, we will focus on the expanded generating functions related to the Fibonacci Substitution as opposed to closed form representations of the sequence. Throughout the paper, we let $f_1 = 1$ be the first term in the Fibonacci Sequence and $F_1 = a$ be the first iteration of the Fibonacci Substitution.

3 Fibonacci Generating Functions

As stated previously, we can look at the generating functions related to the Fibonacci Substitution in multiple ways. We will focus on the one-sided substitutions in two ways, starting with each of their expanded representations.

3.1 Exponents Represent the Positions

Let the exponents represent the positions of the 'a' values in the subwords of the Fibonacci Substitution. By converting the 'a' and 'b' letters into 1's and 0's, a sequence of binary digits is created.

Definition 3.1. *Let a_k be the position of the k^{th} 'a' value in $\lim_{n\to\infty} F_n$. Then a_k is defined as,*

$$a_k = \begin{cases} 1 & \text{if the } k^{th} \text{ letter is 'a'} \\ 0 & \text{if the } k^{th} \text{ letter is 'b'} \end{cases}$$

Through the use of Definition 3.1, the 'a' tiles in the iterations of F_n can be separated from the 'b' tiles. By taking the summation of the 'a' tiles' coefficients, the exponent generating function of the 'a' tiles can be found for each F_n iteration.

Definition 3.2. *Let 'A_n' represent the exponent generating function that corresponds to F_n. Then*

$$A_n(x) = \sum_{k=1}^{f_{n+1}} a_k x^k$$

are the partial sums for the generating function $A(x) = \sum_{k=1}^{\infty} a_k x^k$.

Due to this definition the degrees of the x values in the generating functions represent the positions of the a tiles in the Fibonacci Substitution and each coefficient has a value of 1. The first

8 iterations of $A_n(x)$ are shown below:

$$A_0 = 0$$
$$A_1 = x^1$$
$$A_2 = x^1$$
$$A_3 = x^1 + x^3$$
$$A_4 = x^1 + x^3 + x^4$$
$$A_5 = x^1 + x^3 + x^4 + x^6 + x^8$$
$$A_6 = x^1 + x^3 + x^4 + x^6 + x^8 + x^9 + x^{11} + x^{12}$$
$$A_7 = x^1 + x^3 + x^4 + x^6 + x^8 + x^9 + x^{11} + x^{12} + x^{14} + x^{16} + x^{17} + x^{19} + x^{21}$$

Notice that in A_1 there is $f_1 = \mathbf{1}$ term, in A_2 there is $f_2 = \mathbf{1}$ term and in A_3 there are $f_3 = \mathbf{2}$ terms. This process continues on so that at A_n there are f_n values in the polynomial. Between A_1 and A_3 each F_1 term is multiplied by x^{f_2}, between A_2 and A_4 each F_2 term is multiplied by x^{f_4} and so on so that between step A_{n-1} and A_{n+1} each term in A_{n-1} is multiplied by $x^{f_{n+1}}$. Thus, $A_n(x)$ can be generalized to the following,

Proposition 3.3. ***Exponent Generating Function of the 'a' Tiles*** *For* $n \geqslant 2$*,*

$$A_{n+1} = A_n + (A_{n-1})x^{f_{n+1}}$$

Proof of Proposition 3.3. From Definition 2.3 we know that $F_{n+1} = F_n F_{n-1}$. From Proposition 2.6 we know each F_n has f_n 'a' values and f_{n+1} total terms. We let the coefficients of 'a' tiles be $a = 1$; we let the coefficients of the 'b' tiles be $b = 0$. Then,

$$\begin{aligned} a_1 \dots a_{f_{n+2}} &= F_{n+1} \\ &= F_n F_{n-1} \\ &= a_1 \dots a_{f_{n+1}} a_{f_1} \dots a_{f_n} \end{aligned}$$

Therefore,

$$\begin{aligned} A_{n+1} &= \sum_{k=1}^{f_{n+2}} a_k x^k \\ &= \sum_{k=1}^{f_{n+1}} a_k x^k + \sum_{k=f_{n+1}+1}^{f_{n+2}} a_k x^k \\ &= A_n + (\sum_{k=1}^{f_n} a_k x^k) x^{f_{n+1}} \\ &= A_n + (A_{n-1}) x^{f_{n+1}} \end{aligned}$$

■

If we repeat what we did on the 'a' tiles for the 'b' values we can obtain its exponent generating function.

Definition 3.4. *Let b_k be the position of the k^{th} 'b' value in $\lim_{n\to\infty} F_n$. Then b_k is defined as,*

$$b_k = \begin{cases} 0 & \text{if the } k^{th} \text{ letter is 'a'} \\ 1 & \text{if the } k^{th} \text{ letter is 'b'} \end{cases}$$

Definition 2.3 gives all the terms in the Fibonacci Substitution for F_n. If A_n is subtracted from the total F_n summation of the 'a' and 'b' tiles then only the summation of the 'b' tiles is left. By using this process, the 'b' exponent generating function is as follows,

Definition 3.5. *Let $B_n(x)$ represent the exponent generating function for the b_n tiles in F_n. Then,*

$$B_n(x) = \sum_{k=1}^{f_{n+1}} x^k - A_n(x) = \sum_{k=1}^{f_{n+1}} b_k x^k$$

are the partial sums for the generating function $B(x) = \sum_{k=1}^{\infty} b_k x^k$.

Due to this definition, the first 8 iterations of $B_n(x)$ are:

$$\begin{aligned}
B_0 &= 0 \\
B_1 &= 0 \\
B_2 &= x^2 \\
B_3 &= x^2 \\
B_4 &= x^2 + x^5 \\
B_5 &= x^2 + x^5 + x^7 \\
B_6 &= x^2 + x^5 + x^7 + x^{10} + x^{13} \\
B_7 &= x^2 + x^5 + x^7 + x^{10} + x^{13} + x^{15} + x^{18} + x^{20}
\end{aligned}$$

Notice that the exponents in each $B_n(x)$ have the degrees that were missing in the corresponding $A_n(x)$. Further, $B_n(x)$ follows a similar pattern, but begins a step later than $A_n(x)$. Thus, we have the following recurrence:

Proposition 3.6. ***Exponent Generating Function of the 'b' Tiles*** *For $n \geqslant 3$,*

$$B_{n+1} = B_n + (B_{n-1})x^{f_{n+1}}$$

Proof of Proposition 3.6. By Definition 3.4 and Proposition 3.3,

$$
\begin{aligned}
B_{n+1} &= \sum_{k=1}^{f_{n+2}} x^k - A_{n+1} \\
&= \sum_{k=1}^{f_{n+2}} x^k - (A_n + A_{n-1}x^{f_{n+1}}) \\
&= (\sum_{k=1}^{f_{n+1}} x^k - A_n) + (\sum_{k=f_{n+1}+1}^{f_{n+2}} x^k - A_{n-1}) \\
&= (\sum_{k=1}^{f_{n+1}} x^k - A_n) + (\sum_{k=1}^{f_n} x^k - A_{n-1})x^{f_{n+1}} \\
&= B_n + (B_{n-1})x^{f_{n+1}}
\end{aligned}
$$

■

The generating functions $A_n(x)$ and $B_n(x)$ provide useful representations of the 'a' and 'b' values in the Fibonacci Substitution. By manipulating these generating functions, further information about the Fibonacci Substitution can be gained such as the radius and interval of convergence.

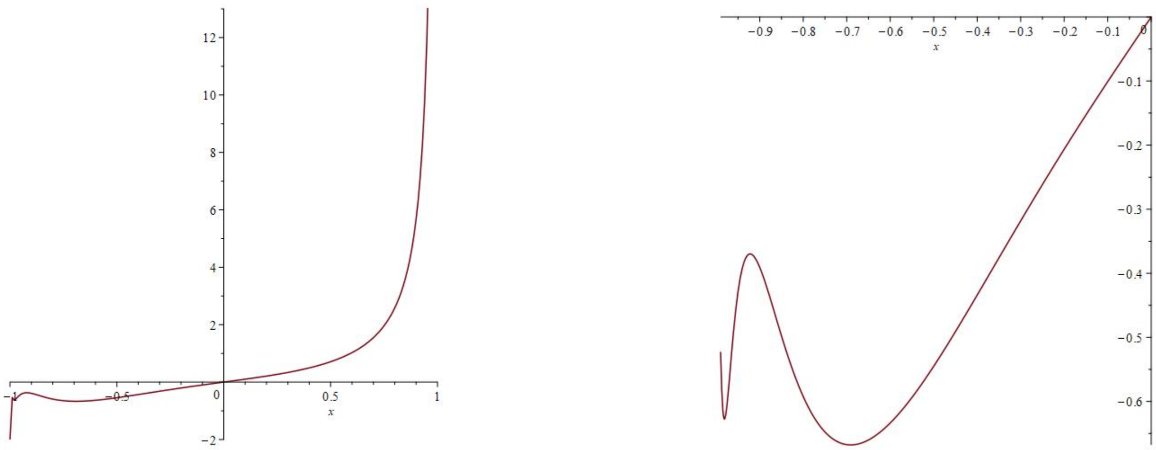
Theorem 3.7. *A(x) and B(x) have radius of convergence of $R = 1$ and converge on the interval $(-1, 1)$.*

Proof of Theorem 3.7. We have $A(x) + B(x) = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Thus on [0,1) we have,

$$0 \leqslant A(x), B(x) \leqslant \sum_{k=1}^{\infty} x^k$$

Since $\sum_{k=1}^{\infty} x^k$ is a geometric sequence, it converges absolutely on $-1 < x < 1$. Thus by the comparison test, A(x) and B(x) must also converge absolutely on $-1 < x < 1$. Clearly A(x) and B(x) diverge at $x = 1$, resulting in a radius of convergence of $R = 1$. ■

The interval and radius of convergence shows that the interesting behaviour of the Fibonacci Substitution occurs on $-1 < x < 1$, allowing focus to be restricted to this interval. As shown below in the following figures, the generating function diverges to infinity when above $x = 1$ and diverges to either infinity or negative infinity depending on the iteration n when below $x = -1$.



(a) Range $[-1, 1]$ (b) Range $[-0.99, 0]$

Figure 1: A_n when $n = 28$

3.2 Coefficients Represent the Positions

Let the coefficients now represent the positions of the ‘a’ values in the subwords of the Fibonacci Substitution.

Definition 3.8. *Let c_k be the position of the k^{th} ‘a’ value in $\lim_{n=\infty} F_n$*

By using Definition 3.8, the ‘a’ tiles in F_n can be separated from the ‘b’ tiles. By taking the summation of the ‘a’ tiles’ coefficients, the coefficient generating function of the ‘a’ tiles can be found for each F_n iteration.

Definition 3.9. *Let ‘C_n’ represent the coefficient generating function that corresponds to F_n. Then,*

$$C_n(x) = \sum_{k=1}^{f_n} c_k x^k$$

are the partial sums for the generating function $C(x) = \sum_{k=1}^{\infty} c_k x^k$.

The generating functions of the first 8 subwords using Definition 3.9 are,

$C_0 = 0$

$C_1 = x^1$

$C_2 = x^1$

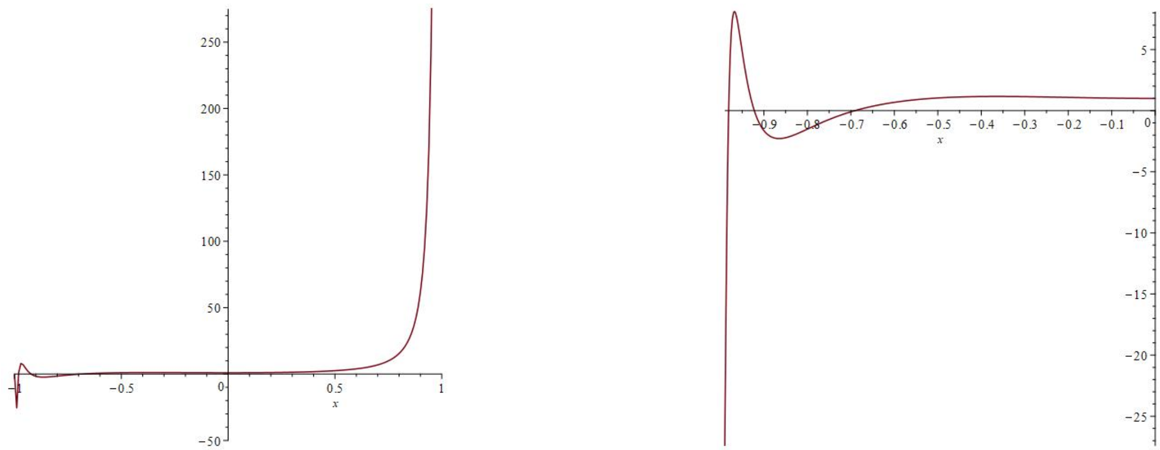
$C_3 = x^1 + 3x^2$

$C_4 = x^1 + 3x^2 + 4x^3$

$C_5 = x^1 + 3x^2 + 4x^3 + 6x^4 + 8x^5$

$C_6 = x^1 + 3x^2 + 4x^3 + 6x^4 + 8x^5 + 9x^6 + 11x^7 + 12x^8$

$C_7 = x^1 + 3x^2 + 4x^3 + 6x^4 + 8x^5 + 9x^6 + 11x^7 + 12x^8 + 14x^9 + 16x^{10} + 17x^{11} + 19x^{12} + 21x^{13}$



(a) Range $[-1, 1]$ (b) Range $[-0.99, 0]$

Figure 2: A'_n when $n = 28$

where C_3 has two 'a' values that are in position 1 and 3, C_4 has three 'a' values in positions 1, 3, 4 and so on.

Notice that for C_3 the new term $3x^2$ is the same as $[2x^1 + x^1]^1$, for C_4 the new term $4x^3$ is the same as $[3x^1 + x^1]^2$, for C_5 the new terms $6x^4 + 8x^5$ is the same as $[5(x^1 + 3x^2) + x^1 + 3x^2]^3$. This pattern continues on and thus we derive the formula,

$$C_{n+1} = C_n + [deg(C_{n+1})(x^1 + x^2 + ... + x^{deg(C_{n-1})}) + C_{n-1}]x^{deg(C_n)}$$

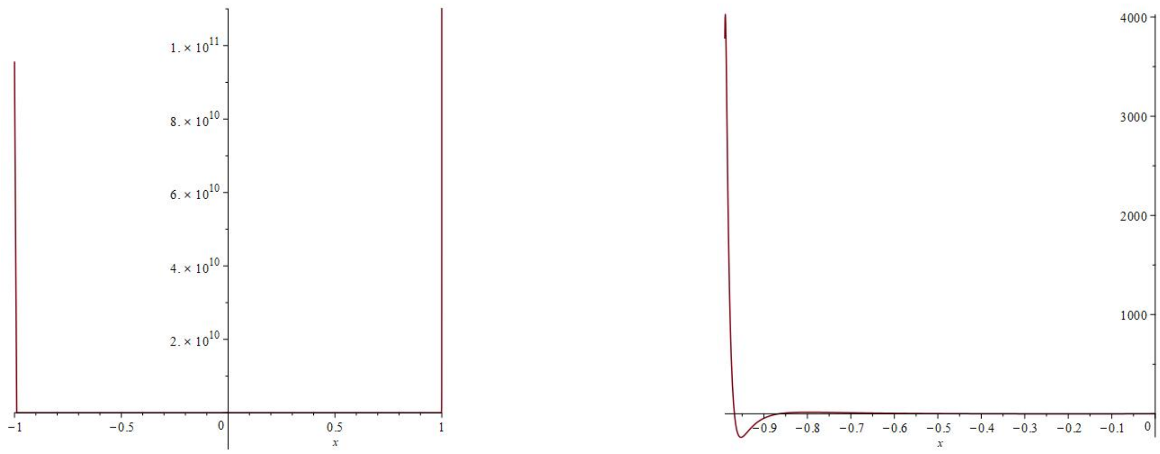
The highest degree in C_{n+1} has a value of the ${f_{n+1}}^{th}$ Fibonacci number, C_n has a degree of the ${f_n}^{th}$ Fibonacci number and C_{n-1} a degree of the ${f_{n-1}}^{th}$ Fibonacci number. This pattern repeats, allowing for the generalization of the coefficient generating function to the following proposition:

Proposition 3.10. ***Coefficient Generating Function of the 'a' Tiles*** *For* $n \geqslant 2$,

$$C_{n+1} = C_n + [f_{n+1}(x^1 + x^2 + ... + x^{f_{n-1}}) + C_{n-1}]x^{f_n}$$

Proof of Proposition 3.10. From Proposition 2.6 we know each F_n has f_n 'a' values and f_{n+1} total terms. Since the positions of the 'a' values in F_{n+1} are $c_1, c_2, \ldots, c_{f_{n+1}}$ and $F_{n+1} = F_n F_{n-1}$ then,

$$c_{k+f_n} = c_k + f_{n+1}$$



(a) Range $[-1, 1]$

(b) Range $[-0.99, 0]$

Figure 3: A_n'' when $n = 28$

for $1 \leqslant k \leqslant f_{n-1}$. Thus,

$$\begin{aligned} C_{n+1} &= \sum_{k=1}^{f_{n+1}} c_k x^k \\ &= \sum_{k=1}^{f_n} c_k x^k + \sum_{k=f_n+1}^{f_{n+1}} c_k x^k \\ &= C_n + (\sum_{k=1}^{f_{n-1}} (c_k + f_{n+1}) x^k) x^{f_n} \\ &= C_n + [f_{n+1}(x^1 + x^2 + ... + x^{f_{n-1}}) + C_{n-1}] x^{f_n} \end{aligned}$$

■

Next we find the coefficient generating functions for the 'b' tiles in the Fibonacci Substitution:

Definition 3.11. *Let d_k be the position of the k^{th} 'b' value in $\lim_{n=\infty} F_n$.*

Definition 3.11 gives an easy way to separate the 'b' values in F_n from the 'a' tiles. Recall that each F_n has f_{n-1} 'b' tiles. By taking the summation of the 'b' tiles' coefficients, the coefficient generating function of the 'b' tiles can be found for each F_n iteration.

Definition 3.12. *Let 'D_n' represent the coefficient generating function that corresponds to F_n. Then,*

$$D_n(x) = \sum_{k=1}^{f_{n-1}} d_k x^k$$

are the partial sums for the generating function $D(x) = \sum_{k=1}^{\infty} d_k x^k$.

The first 8 iteration of $D_n(x)$ can be found using Definition 3.12. They are as follows,

$$\begin{aligned}
D_0 &= 0 \\
D_1 &= 0 \\
D_2 &= 2x^1 \\
D_3 &= 2x^1 \\
D_4 &= 2x^1 + 5x^2 \\
D_5 &= 2x^1 + 5x^2 + 7x^3 \\
D_6 &= 2x^1 + 5x^2 + 7x^3 + 10x^4 + 13x^5 \\
D_7 &= 2x^1 + 5x^2 + 7x^3 + 10x^4 + 13x^5 + 15x^6 + 18x^7 + 20x^8
\end{aligned}$$

Note that the coefficients that are missing in $C_n(x)$ are the coefficients in $D_n(x)$. The iterations of $D_n(x)$ follow a pattern similar to that above, starting a step delayed from $C_n(x)$ with $f_3 = 2$ as the coefficient. This is due to F_1 of the Fibonacci Substitution lacking a 'b' value.

Proposition 3.13. ***Coefficient Generating Function of the 'b' Tiles*** *For $n \geqslant 3$,*

$$D_{n+1} = D_n + (f_{n+1}(x^1 + x^2 + ... + x^{f_{n-2}}) + D_{n-1})x^{f_{n-1}}$$

Proof of Proposition 3.13. From Corollary 2.6 we know each F_n has f_n 'a' values and f_{n+1} total terms. Since the positions of the 'b' values in F_{n+1} are $d_1, d_2, \dots, d_{f_{n+1}}$ and $F_{n+1} = F_n F_{n-1}$ we get,

$$d_{k+f_n} = d_k + f_{n+1}$$

for $1 \leqslant k \leqslant f_{n-1}$. Thus,

$$\begin{aligned}
D_{n+1} &= \sum_{k=1}^{f_{n+1}} d_k x^k \\
&= \sum_{k=1}^{f_{n-1}} d_k x^k + \sum_{k=f_{n-1}+1}^{f_{n+1}} d_k x^k \\
&= D_n + (\sum_{k=1}^{f_{n-2}} (d_k + f_{n+1}) x^k) x^{f_{n-1}} \\
&= D_n + [f_{n+1}(x^1 + x^2 + ... + x^{f_{n-1}}) + D_{n-1}] x^{f_n}
\end{aligned}$$

∎

Proposition 3.14. *The c_k and d_k satisfy the following relations,*

$$c_k + 1 \leqslant c_{k+1} \leqslant c_k + 2 \quad \textit{and} \quad d_k + 1 \leqslant d_{k+1} \leqslant d_k + 3$$

Proof of Proposition 3.14. It is clear that $c_{k+1} \geqslant c_k + 1$, since position c_{k+1} must occur after position c_k. From Proposition 2.4, no two 'b' values can occur successively. Therefore c_{k+1} must occur in either at $c_k + 1$ or $c_k + 2$.

From Proposition 2.5 we know that there can not be three successive 'a' tiles. Therefore the d_k equation follows similarly. ∎

Proposition 3.15. *The c_k and d_k satisfy the following relations,*

$$k \leqslant c_k \leqslant 2k \quad and \quad k \leqslant d_k \leqslant 3k$$

Proof of Proposition 3.15. Let $k = 1$, $k \in Z$. We know c_1 is in position 1. Thus we get,

$$k \leqslant c_1 \leqslant 2k$$
$$1 \leqslant 1 \leqslant 2, \text{ which is true.}$$

Let $k = n + 1$, $k, n \in Z$. Assume that the relation $k \leqslant c_k \leqslant 2k$ holds. Then using Proposition 3.14 we get,

$$n + 1 \leqslant c_n + 1 \leqslant c_{n+1} \leqslant c_n + 2 \leqslant 2n + 2$$

By induction the result holds. The d_k follows similarly. ∎

Theorem 3.16. *C(x) and D(x) converge on the interval $(-1, 1)$ with radius of convergence $R = 1$.*

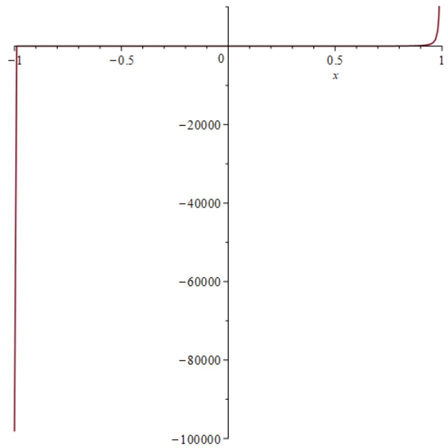
Proof of Theorem 3.16. We want to prove that the Fibonacci Substitution converges on $(-1, 1)$. From Proposition 3.15, we know that $k \leqslant c_k \leqslant 2k$ where k is the number of positions and c_k is the k^{th} a value. The comparison test then gives,

$$\sum_{k=1}^{\infty} k|x|^k \leqslant \sum_{k=1}^{\infty} c_k|x|^k \leqslant \sum_{k=1}^{\infty} 2k|x|^k$$

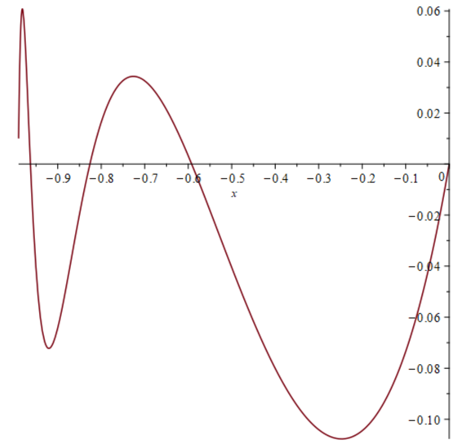
Thus on $[0, 1)$,

$$\sum_{k=1}^{\infty} c_k|x|^k \leqslant \sum_{k=1}^{\infty} 2k|x|^k = 2x \sum_{k=1}^{\infty} kx^{k-1} = 2x \left(\frac{1}{1-x}\right)'$$

Thus $\sum_{k=1}^{\infty} c_k x^k$ absolutely converges on $(-1, 1)$. Clearly $C(x)$ diverges at $x = 1$, and so $R = 1$. $D(x)$ follows similarly. ∎



(a) Range $[-1, 1]$



(b) Range $[-0.99, 0]$

Figure 4: C_n when $n = 27$

4 Zeros and Critical Points

4.1 Exponent Generating Function

As seen in the graph of A_{27}, the polynomial begins to oscillate above and below the x-axis as it approach $x = -1$. As n is increased, the number of oscillations seems to also increase, implying that the number of zeroes must also increase. To quantify this increase, python code was used to generate the A_n polynomials for n values ranging from $1 - 21$ along with their first and second derivatives. Once $n > 21$, A_n became too large for the computer to calculate the zeroes. The real roots of these polynomials were then calculated along with the $\lim_{x\to-1}$. The following table displays the results,

Table 1: Zeroes and limits at -1 for A_n and its derivatives

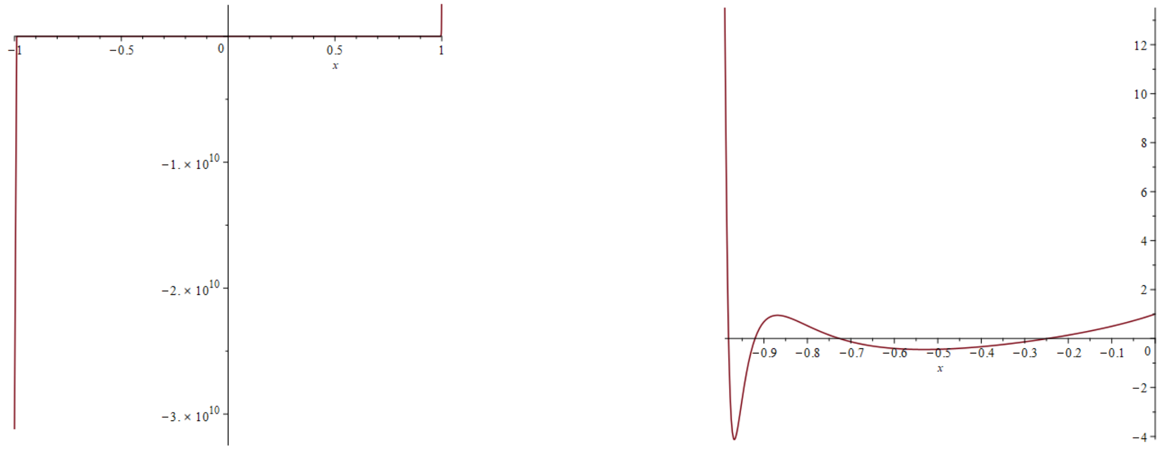
n	A_n zeroes	$\lim_{x\to-1} A_n$	A'_n zeroes	$\lim_{x\to-1} A'_n$	A''_n zeroes	$\lim_{x\to-1} A''_n$
1	0	$-$	D.N.E.	$+$	$x \in R$	0
2	0	$-$	D.N.E.	$+$	$x \in R$	0
3	0	$-$	0	$+$	0	$-$
4	-1.465571232 0	$+$	-0.75 0	$-$	-0.5 0	$+$
5	0	$+$	-0.5 0	$-$	-0.5902396386 0	$+$
6	-1 0	$+$	-0.7025894375	$-$	-0.3645389694 0	$+$

7	0	−	-0.6223818629 -0.2474040974	+	-0.8580198336 -0.3644230629 0	−
8	0	−	-0.9294403565 -0.6908129137]	+	-0.8708059914 -0.3644229720 0	−
9	0	−	-0.9200122514 -0.6908241600	+	-0.8644109919 -0.3644229720 0	−
10	0	+	-1 -0.9219017110 -0.6908241523	−	-0.9843296329 -0.8651360980 -0.3644229720 0	+
11	-0.9943626909 0	+	-0.9780486735 -0.9221399149 -0.6908241523	−	-0.9654659749 -0.8651617552 -0.3644229720 0	+
12	-1 0	+	-0.9823056016 -0.9221354447 -0.6908241523	−	-0.9682831429 -0.8651517404 -0.3644229720 0	+
13	0	−	-0.9953740374 -0.9812092860 -0.9221354500 -0.6908241523	+	-0.9919497760 -0.9678704244 -0.8651517404 -0.3644229720 0	−
14	0	−	-0.9959759803 -0.9810974541 -0.9221354500 -0.6908241523	+	-0.9927761863 -0.9678614762 -0.8651517404 -0.3644229720 0	−
15	0	−	-0.9953791985 -0.9810994712 -0.9221354500 -0.6908241523	+	-0.9923253509 -0.9678614874 -0.8651517404 -0.3644229720 0	−
16	0	+	-1 -0.9954991322 -0.9810994687 -0.92213554500 -0.6908241523	−	-0.9991430883 -0.9923825770 -0.9678614874 -0.8651517404 -0.3644229720 0	+

17	-0.9996853705 0	$+$	-0.9987650920 -0.9955145129 -0.9810994687 -0.9221354500 -0.6908241523	$-$	-0.9980601292 -0.9923841822 -0.9678614874 -0.8651517404 -0.3644229720 0	$+$
18	-1 0	$+$	-0.9990087474 -0.9955142179 -0.9810994687 -0.9221354500 -0.6908241523	$-$	-0.9982261951 -0.9923841799 -0.9678614874 -0.8651517404 -0.3644229720 0	$+$
19	0	$-$	-0.9997416509 -0.9989459271 -0.9955142183 -0.9810994687 -0.9221354500 -0.6908241523	$+$	-0.9995507757 -0.9982012043 -0.9923841799 -0.9678614874 -0.8651517404 -0.3644229720 0	$-$
20	0	$-$	-0.9997755003 -0.9989394742 -0.9955142183 -0.9810994687 -0.9221354500 -0.6908241523	$+$	-0.9995972418 -0.9982006361 -0.9923841799 -0.9678614874 -0.8651517404 -0.3644229720 0	$-$
21	0	$-$	-0.9997420108 -0.9989395917 -0.9955142183 -0.9810994687 -0.9221354500 -0.6908241523	$+$	-0.9995718027 -0.9982006369 -0.9923841799 -0.9678614874 -0.8651617404 -0.3644229720 0	$-$

A sign pattern for the $\lim_{x\to -1}$ emerges in Table 1, where the A_n polynomials form a $-,-,-,+,+,+$ pattern, A_n' repeats $+,+,+,-,-,-$, and A_n'' has a $-,-,-,+,+,+$ pattern once $n > 6$. A_n' also always has the opposite sign to the A_n polynomials and A_n'' always has the same signs as A_n (once $n > 2$). Through this information, a recurrence relation for the sign of A_n can be found. Recall from Proposition 3.3 that A_n has formula $A_{n+1} = A_n + (A_{n-1})x^{f_{n+1}}$. We can see that the value of A_{n+1} at $x = -1$ is given recursively,

$$A_{n+1}(-1) = A_n(-1) + A_{n-1}(-1)(-1)^{f_{n+1}}$$



(a) Range $[-1, 1]$

(b) Range $[-0.99, 0]$

Figure 5: C_n' when $n = 27$

which explains this oscillating behaviour. From this finding we make the following conjecture,

Conjecture 4.1. *$A(x)$ has an infinite number of zeroes in the interval $-1 < x < 1$.*

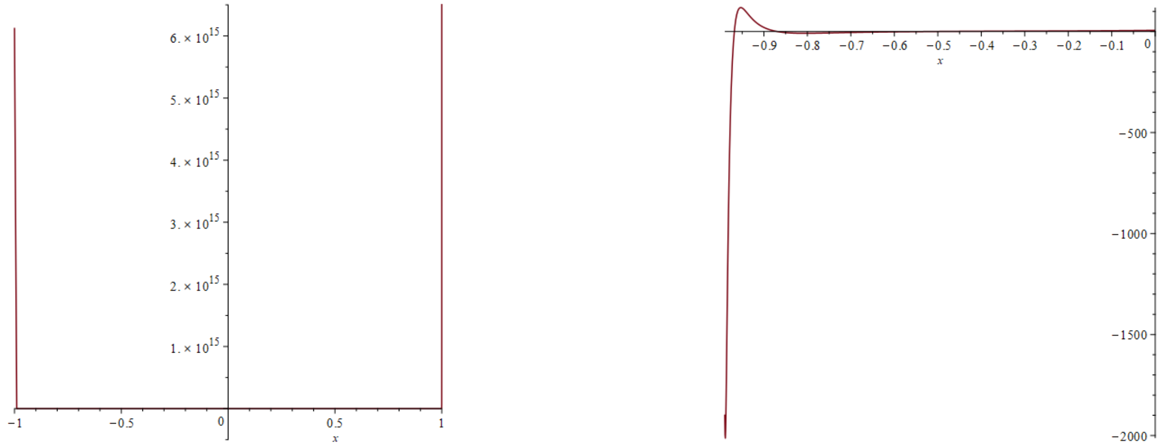
As further evidence, Table 2 depicts the first 50 'a' tiles in the Fibonacci Substitution as encoded by $A_n(x)$, the degrees corresponding to these 'a' terms using A_n, and the signs of each 'a' term (positive or negative). The table does the same for the first derivative and the second derivative, finding their first 50 'a' values and their respective signs.

The signs of the largest degrees for the 'a' values display a pattern that has groups of three positive or negative values and groups of two positive or negative values. By colouring the groups of three identical signs pink and the groups of two signs blue, the Fibonacci Substitution arises again. The pink sign groups can not have more than two groups in a row, representing the 'a' values in the Fibonacci Substitution; likewise, the blue blocks can not occur successively, representing the 'b' tiles in the Fibonacci Substitution.

Table 2: Signs of the first 50 terms of $A(-1), A'(-1), A''(-1)$

	k for $a_k \neq 0$	$(-1)^k$	Non-zero $A(x)'$ degree	$(-1)^k$	Non-zero $A(x)''$ degree	$(-1)^k$
1	1	$-$	0	$+$	1	$-$
2	3	$-$	2	$+$	2	$+$
3	4	$+$	3	$-$	4	$+$
4	6	$+$	5	$-$	6	$+$
5	8	$+$	7	$-$	7	$-$
6	9	$-$	8	$+$	9	$-$
7	11	$-$	10	$+$	10	$+$
8	12	$+$	11	$-$	12	$+$
9	14	$+$	13	$-$	14	$+$

10	16	+	15	–	15	–
11	17	–	16	+	17	–
12	19	–	18	+	19	–
13	21	–	20	+	20	+
14	22	+	21	–	22	+
15	24	+	23	–	23	–
16	25	–	24	+	25	–
17	27	–	26	+	27	–
18	29	–	28	+	28	+
19	30	+	29	–	30	+
20	32	+	31	–	31	–
21	33	–	32	+	33	–
22	35	–	34	+	35	–
23	37	–	36	+	36	+
24	38	+	37	–	38	+
25	40	+	39	–	40	+
26	42	+	41	–	41	–
27	43	–	42	+	43	–
28	45	–	44	+	44	+
29	46	+	45	–	46	+
30	48	+	47	–	48	+
31	50	+	49	–	49	–
32	51	–	50	+	51	–
33	53	–	52	+	53	–
34	55	–	54	+	54	+
35	56	+	55	–	56	+
36	58	+	57	–	57	–
37	59	–	58	+	59	–
38	61	–	60	+	61	–
39	63	–	62	+	62	+
40	64	+	63	–	64	+
41	66	+	65	–	65	–
42	67	–	66	+	67	–
43	69	–	68	+	69	–
44	71	–	70	+	70	+
45	72	+	71	–	72	+
46	74	+	73	–	74	+
47	76	+	75	–	75	–
48	77	–	76	+	77	–
49	79	–	78	+	78	+
50	80	+	79	–	80	+



(a) Range $[-1, 1]$

(b) Range $[-0.99, 0]$

Figure 6: C_n'' when $n = 27$

The leading terms of A_n at $x = -1$ in Table 2 match the signs of the overall A_n polynomials in Table 1. For instance, when $n = 2$ in Table 1 the leading term is x^1, which is negative at $x = -1$. When $n = 3$, Table 1 has $x^1 + x^3$, which is also negative at $x = -1$. The third ‘a’ in Table 2 has the leading term x^4 and when $n = 4$ in Table 1 the equation is $x^1 + x^3 + x^4$: both have a positive sign at $x = -1$. Thus, it seems that the largest degree in A_n may dominate, making the sign of the polynomial at $x = -1$ the same as the leading degree.

Table 3 depicts the first 50 ‘b’ tiles in the Fibonacci Substitution as encoded by $B(x)$, the degrees corresponding to these ‘b’ terms using B_n and the signs of each ‘b’ term (positive or negative). The table does the same for the first derivative and the second derivative, finding their first 50 ‘b’ values and their respective signs.

Table 3: Signs of the first 50 terms of $B(-1), B'(-1), B''(-1)$

	k for $b_k \neq 0$	$(-1)^k$	Non-zero $B(x)'$ degree	$(-1)^k$	Non-zero $B(x)''$ degree	$(-1)^k$
1	2	$+$	1	$-$	0	$+$
2	5	$-$	4	$+$	3	$-$
3	7	$-$	6	$+$	5	$-$
4	10	$+$	9	$-$	8	$+$
5	13	$-$	12	$+$	11	$-$
6	15	$-$	14	$+$	13	$-$
7	18	$+$	17	$-$	16	$+$
8	20	$+$	19	$-$	18	$+$
9	23	$-$	22	$+$	21	$-$

10	26	+	25	−	24	+
11	28	+	27	−	26	+
12	31	−	30	+	29	−
13	34	+	33	−	32	+
14	36	+	35	−	34	+
15	39	−	38	+	37	−
16	41	−	40	+	39	−
17	44	+	43	−	42	+
18	47	−	46	+	45	−
19	49	−	48	+	47	−
20	52	+	51	−	50	+
21	54	+	53	−	52	+
22	57	−	56	+	55	−
23	60	+	59	−	58	+
24	62	+	61	−	60	+
25	65	−	64	+	63	−
26	68	+	67	−	66	+
27	70	+	69	−	68	+
28	73	−	72	+	71	−
29	75	−	74	+	73	−
30	78	+	77	−	76	+
31	81	−	80	+	79	−
32	83	−	82	+	81	−
33	86	+	85	−	84	+
34	89	−	88	+	87	−
35	91	−	90	+	89	−
36	94	+	93	−	92	+
37	96	+	95	−	94	+
38	99	−	98	+	97	−
39	102	+	101	−	100	+
40	104	+	103	−	102	+
41	107	−	106	+	105	−
42	109	−	108	+	107	−
43	112	+	111	−	110	+
44	115	−	114	+	113	−
45	117	−	116	+	115	−
46	120	+	119	−	118	+
47	123	−	122	+	121	−
48	125	−	124	+	123	−
49	128	+	127	−	126	+
50	130	+	129	−	128	+

The Fibonacci Substitution arises again in the signs of the leading ‘b’ degrees as encoded by $B(x)$. Signs that are identical and successive are grouped together. We colour the groups of 2 identical signs pink and the signs without a repeat blue. The pink blocks can only have two different groups in a row, representing the ‘a’ terms in the Fibonacci Substitution. The blue blocks can not have another blue block preceding or following another blue block, much like how no two ‘b’ terms can appear successively in the Fibonacci Substitution. Thus, the blue blocks depict the ‘b’ terms.

Again, the leading term for the iterations of B_n dominates, making its sign the sign of the whole iteration. Since the Fibonacci Substitution is infinite and aperiodic, the leading term must constantly change between an even and odd value. In Table 3 and Table 2, odd leading exponents correspond to a negative sign and even leading exponents correspond to a positive sign at $x = -1$. Altogether, we conjecture that this finding means that the Fibonacci Substitution must continue switching signs, causing an infinite amount of zeroes on the interval of $(-1, 1)$.

Conjecture 4.2. *$B(x)$ has an infinite number of zeroes in the interval $-1 < x < 1$.*

4.2 Coefficient Generating Function

Similar to A_n, python code was used to generate C_n along with its first and second derivatives for n values ranging from 1 - 21. The the real roots of these polynomials and their $\lim_{x\to-1}$ were calculated. The following table displays the results,

Table 4: Zeroes and Limits at -1 for C_n and its Derivatives

n	C_n zeroes	$\lim_{x\to-1} C_n$	C'_n zeroes	$\lim_{x\to-1} C'_n$	C''_n zeroes	$\lim_{x\to-1} C''_n$
1	0	$-$	D.N.E.	$+$	$x \in R$	0
2	0	$-$	D.N.E.	$+$	$x \in R$	0
3	-0.3333333333 0	$+$	-0.1666666667	$-$	D.N.E.	$+$
4	0	$-$	D.N.E	$+$	-0.25	$-$
5	0	$-$	-0.3964341172 -0.2669600823	$+$	-0.3358396550	$-$
6	-0.5288497136 0	$+$	-0.2469231852	$-$	D.N.E.	$+$
7	0	$-$	-0.6223818629 -0.2474040974	$+$	-0.4957039445	$-$
8	-0.7483226352 -0.5926087851 0	$-$	-0.6893687201 -0.2474029681	$+$	-0.5298364406	$-$

9	-0.5920920156 0	+	-0.7990766519 -0.7307313380 -0.2474029681	−	-0.7737569351 -0.5317091164	+
10	-0.8246650504 -0.5920229150 0	−	-0.7261838575 -0.2474029681	+	-0.5317068314	−
11	-0.8260305973 -0.5920229150 0	−	-0.7261956250 -0.2474029681	+	-0.5317068314	−
12	-0.9473504187 -0.8260342287 -0.5920929150 0	+	-0.9185999767 -0.7261956256 -0.2474029681	−	-0.8687584956 -0.5317068314	+
13	-0.8260342285 -0.5920929150 0	−	-0.9603993697 -0.9204001421 -0.7261956256 -0.2474029681	+	-0.9520878626 -0.8687712015 -0.5317068314	−
14	-0.9774497434 -0.9627967299 -0.8260342285 -0.5920929150 0	−	-0.9729465578 -0.9203964040 -0.7261956256 -0.2474029681	+	-0.9646975098 -0.8687712013 -0.5317068314	−
15	-0.9627488702 -0.8260342285 -0.5920929150 0	+	-0.9203964040 -0.7261956256 -0.2474029681	−	-0.9796106714 -0.9680687558 -0.8687712013 -0.5317068314	+
16	-0.9892864876 -0.9627488811 -0.8260342285 -0.5920929150 0	−	-0.9812760667 -0.9203964040 -0.7261956256 -0.2474029681	+	-0.9680474092 -0.8687712013 -0.5317068314	−
17	-0.9906803792 -0.9627488811 -0.8260342285 -0.5920929150 0	−	-0.98128331111 -0.9203964040 -0.7261956256 -0.2474029681	+	-0.9680474097 -0.8687712013 -0.5317068314	−
18	-0.9959733996 -0.9907083202 -0.9627488811 -0.8260342285 -0.5920929150 0	+	-0.9948198154 -0.9812833113 -0.9203964040 -0.7261956256 -0.2474029681	−	-0.9923722712 -0.9680474097 -0.8687712013 -0.5317068314	+

19	-0.9907083155 -0.9627488811 -0.8260342285 -0.5920929150 0	$-$	-0.9968952456 -0.9955144989 -0.9812833113 -0.9203964040 -0.7261956256 -0.2474029681	$+$	-0.9965135013 -0.9923944086 -0.9680474097 -0.8687712013 -0.5317068314	$-$
20	-0.9981786693 -0.9978665118 -0.9907083155 -0.9627488811 -0.8260342285 -0.5920929150 0	$-$	-0.9980507976 -0.9955079048 -0.9812833113 -0.9203964040 -0.7261956256 -0.2474029681	$+$	-0.9976609630 -0.9923944081 -0.9680474097 -0.8687712013 -0.5317068314	$-$
21	-0.9978211991 -0.9907083155 -0.9627488811 -0.8260342285 -0.5920929150 0	$+$	-0.9955079046 -0.9812833113 -0.9203964040 -0.7261956256 -0.2474029681	$-$	-0.9984426636 -0.9982338637 -0.9923944081 -0.9680474097 -0.8687712013 -0.5317068314	$+$

As Table 4 shows, a pattern for the signs of C_n as $\lim_{x\to -1}$ emerges where C_n repeats signs $-,-,+$, C_n' has $+,+,-$, and C_n'' has $-,-,+$ once $n>3$. Also, notice that for any of the $-$ signs there are always an odd amount of zeroes and as n increases the quantity of these zeroes must either stay the same as the previous negative C_n's amount of zeroes, or increase by 2. On the other hand, the $+$ signs always have an even amount of zeroes; and similar to before, the quantity of zeroes must be the same as the previous iteration with a $+$ sign or increase by 2 to the next even number. This result suggests that the number of roots can not decrease, leading to our following conjecture,

Conjecture 4.3. *$C(x)$ has an infinite number of zeroes in the interval $-1<x<1$.*

Similar to $C(x)$ we posit that $D(x)$ has an infinite number of zeroes on $(-1,1)$; however, analysis of its roots was not carried out.

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