Characterizing Semi-Dirichlet Algebras and their Graphs

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Abstract
In this paper, we relate certain finite-dimensional operator algebras to graphs and define their equivalent properties. Using these graph's properties, we will determine which algebras are semi-Dirichlet and how many non-unitarily similar semi-Dirichlet algebras there are corresponding to graphs with up to five vertices.

1 Introduction

A C⋆-algebra is an algebra with a norm structure and an involution. These algebras do not need to be unital and are not necessarily commutative. See Murphy **[4]** for more information on C⋆-algebras. The kind of algebra we are interested in is an operator algebra. This is a subalgebra of a C⋆-algebra, which may not be closed under involution. Dirichlet and semi-Dirichlet algebras are subalgebras of these C⋆-algebras with certain properties. Dirichlet algebras originate from the Dirichlet problem in complex analysis. It asks if any continuous function on the boundary of a region such as $\mathbb{T} = \{z \in \mathbb{C}|\ |z| = 1\}$ can be extended to a harmonic function on the interior of that region $\mathbb{D} = \{z \in \mathbb{C}|\ |z| < 1\}$. See **[6]** for a version of the original Dirichlet problem. The Dirichlet problem can be restated to ask if

$$\overline{A(\mathbb{D}) + A(\mathbb{D})^\star} = C(\mathbb{T}),$$

where $A(\mathbb{D})$ is the disk algebra, which is the collection of all continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$, and $C(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$. Gleason **[3]** defined the Dirichlet property for commutative subalgebras in a general setting. Arveson **[1]** extended Gleason's work from a commutative setting to include non-commutative algebras. Davidson and Katsoulis **[2]** went on to define the semi-Dirichlet property of algebras in a general setting. A Dirichlet algebra $\mathcal{A}$ is one that satisfies

$$\mathcal{A} \subseteq \mathcal{C} \text{ is Dirichlet if } \overline{\mathcal{A} + \mathcal{A}^\star} = \mathcal{C},$$

where the overline denotes the closure in the norm of the $C^\star$-algebra. A semi-Dirichlet algebra satisfies the weaker condition that

$$\mathcal{A}^\star\mathcal{A} \subset \overline{\mathcal{A} + \mathcal{A}^\star},$$

where $\mathcal{C}$ denotes a C⋆-algebra and $\mathcal{A}$ denotes a subalgebra. These properties will be described fully in the next sections.

In this paper, we will be working with finite-dimensional subalgebras as they are more simple to study in this setting than in a fully generalized setting. Restricting ourselves to finite dimensions

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means we do not need to include the closure, this restriction also allows for the study of all algebras related to graphs up to unitary similarity.

The goal of this project is to illustrate which subalgebras of M_n are semi-Dirichlet. We will do this by relating algebras to graphs and determining which algebraic properties have equivalent graphical properties. Using these graph properties we will determine which algebras are semi-Dirichlet and how many there are up to unitary similarity. We will briefly summarize Burnside's Theorem and explain how it can be applied to matrices and what that means for their respective graphical representations.

2 Graphs and Algebras

Definition 2.1. *An algebra $\mathcal{A}$ (over $\mathbb{C}$) is a $\mathbb{C}$-vector space with a multiplication satisfying the vector space and ring axioms, and such that for all scalars $\lambda \in \mathbb{C}$ and $a, b \in \mathcal{A}$ such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$.*

The algebras we will be using are subspaces of $M_n = M_n(\mathbb{C}) = \{n \times n$ matrices in $\mathbb{C}\}$. M_n is the prototypical finite-dimensional $C^\star$-algebra. The $\star$-structure will be discussed below.

Definition 2.2. *A subalgebra of an algebra, is a vector subspace and a subring closed under multiplication.*

We can say that M_n = span$\{E_{ij} |\ 1 \leq i, j \leq n\}$, where

$$E_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

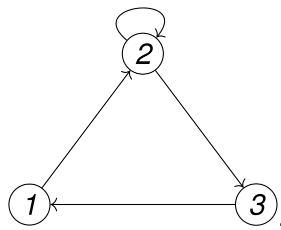
has a 1 in the ij entry and 0 everywhere else. This has the multiplication rule:

$$E_{ij}E_{k\ell} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$
$$= \begin{cases} E_{i\ell}, \text{ if } j = k \\ 0, \text{ if } j \neq k \end{cases}.$$

With this notation, if $S \subseteq \{(i, j) |\ 1 \leq i, j \leq n\}$ is any subset of pairs of indices, then we can define the subspace $\mathcal{A}_S$ = span$\{E_{ij} |\ (i, j) \in S\}$. This leads to the question of which A_S are subalgebras. To answer that question, we will look at graphical representations of A_S.

A directed graph is a graph where edges are arrows and loops are allowed. Let G be a graph on n vertices $\{1, 2, \cdots, n\}$, where $i \to j$ means that there exists an edge from i to j.

Example 2.3. *Below is an example of a directed graph:*



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This is a graph on three vertices with edges $1 \to 2, 2 \to 2, 2 \to 3,$ *and* $3 \to 1$*.*

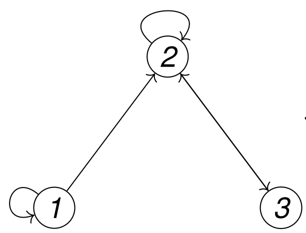
These graphs can be used to represent subspaces of M_n. These represent size in a matrix with vertices of the graph and the edges of the graph fill in the entries. This is shown in the following example.

Definition 2.4. *Suppose* G *is a directed graph on n vertices. Define the edge set* $S \subseteq \{(i, j)|\ 1 \leq i, j \leq n\}$*, where* $(i, j) \in S$ *if and only if* $i \to j$ *is an edge in* G*. Then* $A_G := A_S$*.*

Using this process, we can construct a matrix representation of our graphs denoted as A_G. Matrix representations preserve certain graph properties, and these properties will appear as an equivalent matrix property.

Example 2.5. *Let*

$G =$



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From the graph we get that:

$$A_G = \textit{span}\{E_{11}, E_{12}, E_{22}, E_{23}, E_{32}\} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & d \\ 0 & e & 0 \end{pmatrix} \middle| \ a, b, c, d, e \in \mathbb{C} \right\}.$$

The notation we will be using for this paper will be representing A_G as a matrix in which stars denote which elements in A_G can be nonzero; such a matrix will be referred to as the star matrix. With this notation from the previous example, we get that

$$A_G = \begin{pmatrix} \star & \star & 0 \\ 0 & \star & \star \\ 0 & \star & 0 \end{pmatrix},$$

is the set of all matrices of this form where the stars represent any scalars. Using the example above it is clear that this A_G is not a subalgebra as

$$E_{12}E_{23} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_{13} \notin A_G,$$

thus A_G is not closed under multiplication. The next proposition will show when an A_G is a subalgebra. To make the previous A_G example a subalgebra, both E_{13} and E_{33} would need to be added to A_G. This subalgebra can be written as

$$\mathcal{A}_{TC(G)} = A_G + \text{span}\{E_{13}, E_{33}\} = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix},$$

and $TC(G)$ is the transitive closure of G, see Corollary 2.7.

Proposition 2.6. *Let G be a directed graph on n vertices, then:*

(i) *A_G is a subalgebra if and only if G is transitive.*

(ii) *A_G contains the identity matrix I if and only if G is reflexive.*

Proof. (i) ($\Leftarrow$) Let G be transitive. We need to show that A_G is closed under multiplication. By distributivity, it suffices to show that the product of any two basis vectors in A_G is again in A_G. To this end, let E_{ij} and E_{kl} be in A_G, which means $i \to j$ and $k \to l$ are edges in G. Using the multiplication rule unless $j = k$ which gives $E_{ij}E_{kl} = E_{il}$. In this latter case, by transitivity since $j = k$, then $i \to l$ is an edge in G. Thus E_{il} is in A_G, so A_G is closed under multiplication and is a subalgebra.

(i) ($\Rightarrow$) Let A_G be a subalgebra, then A_G is closed under multiplication. Assume $i \to j$ and $j \to k$ are edges in G. Hence, $E_{ij}E_{jk}$ are in A_G, and so $E_{ik} = E_{ij}E_{jk}$ is in A_G. Therefore, $i \to k$ is an edge in G. Thus G is transitive.

(ii) ($\Leftarrow$) Let G be reflexive, then $i \to i, j \to j, \ldots, n \to n$ are edges in G then

$$A_G \supseteq \begin{pmatrix} \star & 0 & \cdots & 0 \\ 0 & \star & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \star \end{pmatrix},$$

notice that $I \in A_G$, thus A_G contains the identity matrix.

(ii) ($\Rightarrow$) Let A_G contain I, then

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \in A_G.$$

This means that $E_{11}, E_{22}, \ldots, E_{nn}$ are in A_G, which means $1 \to 1, 2 \to 2, \ldots, n \to n$ are edges in G, thus G is reflexive.

□

Corollary 2.7. *The subalgebra generated by A_G is*

$$\text{alg}(A_G) = A_{TC(G)}, \text{ where } TC(G) \text{ is the Transitive closure of } G.$$

In particular, C$^\star(A_G)$ is M_n if and only if $TC(G)$ is the complete graph on n vertices, which occurs if and only if G is weakly connected. Meaning the corresponding undirected graph $G \cup G^\star$ is connected.

The subalgebras we are interested in are star-subalgebras, which are a more restrictive class of subalgebras.

Definition 2.8. *For all $A \in M_n$, $A^\star$ is the conjugate transpose or adjoint:*

$$A = (a_{ij}) \Rightarrow A^\star = (\overline{a_{ji}}),$$

where $\overline{a_{ji}}$ is complex conjugation.

Example 2.9. *If* $A = \begin{pmatrix} 1 & i \\ 3-i & 2i \end{pmatrix}$, *then* $A^\star = \begin{pmatrix} 1 & i \\ 3-i & 2i \end{pmatrix}^\star = \begin{pmatrix} \overline{1} & \overline{i} \\ \overline{3-i} & \overline{2i} \end{pmatrix}^T = \begin{pmatrix} 1 & 3+i \\ -i & -2i \end{pmatrix}.$

Example 2.10. *If* $A = \begin{pmatrix} 0 & i \\ -i & 1 \end{pmatrix}$, *then* $A^\star = \begin{pmatrix} 0 & i \\ -i & 1 \end{pmatrix}^\star = \begin{pmatrix} 0 & i \\ -i & 1 \end{pmatrix}$, *here $A = A^\star$, so A is called self-adjoint.*

Definition 2.11. *A star-subalgebra is a subalgebra that is closed under taking $\star$'s.*

The basic properties of the star operation are

(i) $(A+B)^\star = A^\star + B^\star$,

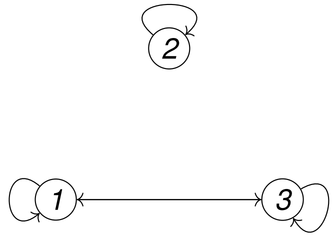
(ii) $(\lambda A^\star) = \overline{\lambda} A^\star$,

(iii) $(AB)^\star = B^\star A^\star$,

(iv) $(A^\star)^\star = A$.

Example 2.12. *Let*

$$A = \begin{pmatrix} \star & 0 & \star \\ 0 & \star & 0 \\ \star & 0 & \star \end{pmatrix}$$



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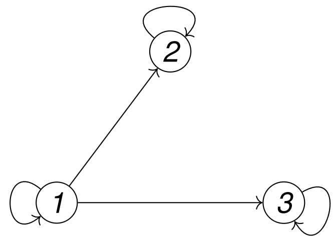
A *is a star-subalgebra.*
Let

$$B = \begin{pmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$$



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B *is an algebra, but is not a star-subalgebra.*

The $\star$-closure property of algebras translates to the graphs as symmetry.

Proposition 2.13. *The following are equivalent:*

(i) A_G *is closed under taking* $\star$*'s.*

(ii) *The star matrix of* A_G *is symmetric.*

(iii) *The graph* G *is symmetric*

Proof. $(ii) \Rightarrow (i)$ If A_G has a symmetric star matrix, then if E_{ij} is in A_G, then E_{ji} is also in A_G. Because $E_{ij}^\star = E_{ji}$ it follows by linearity A_G is closed under taking $\star$'s.

$(i) \Rightarrow (iii)$ If A_G is closed under taking $\star$'s, then if E_{ij} is in A_G, then E_{ji} is also in A_G. So if $i \to j$ is an edge in G, then $j \to i$ is an edge in G. Thus G is symmetric.

$(iii) \Rightarrow (ii)$ If G is symmetric, then if $i \to j$ is an edge in G, then $j \to i$ is also an edge in G. So if E_{ij} is in A_G, then E_{ji} is in A_G. Thus A_G will have a symmetric star matrix. □

Definition 2.14. *If* $S \subseteq M_n$*, the star-subalgebra (*$C^\star$*-subalgebra) generated by* S *is:*

$$\begin{aligned} C^\star(S) &= \bigcap \{A \subseteq M_n |\ A \text{ is a } C^\star\text{-subalgebra, } A \supseteq S\} \\ &= \text{span}\{a_1, \cdots, a_n |\ a_1, \cdots, a_n \in S \cup S^\star, n \geq 1\} \end{aligned}$$

Corollary 2.15. *If* G *is a graph then* $C^\star(A_G) = A_H$*, where* H *is the symmetric and transitive closure of* G*. That is,*

$$H = TC(G \cup \tilde{G}),$$

where $\tilde{G}$ *represents the graph* G *with reversed edges.*

Example 2.16. *Let*

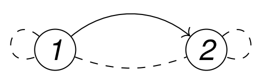
$$T_n = \begin{pmatrix} \star & \cdots & \cdots & \star \\ 0 & \star & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \star \end{pmatrix} = \{A \in M_n |\ A \text{ is upper triangular}\},$$

then $T_n^\star = \{A \in M_n |\ A$ *is lower triangular*$\}$*. Thus,* $C^\star(T_n) \supseteq T_n, T_n^\star$*, which implies that* $C^\star(T_n) = T_n + T_n^\star = M_n$*.*

Example 2.17. *Let* $A = \begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix} =$ *span*$\{E_{12}\}$*, and so* $C^\star(A)$ *contains* E_{12} *and* $E_{12}^\star$*. Hence,*

$$\left.\begin{aligned} E_{12}^\star &= E_{21} \\ E_{12}E_{21} &= E_{11} \\ E_{21}E_{12} &= E_{22} \end{aligned}\right\} \text{ are in } C^\star(A) = M_2.$$

This means that $C^\star(A)$ *contains* $E_{11}, E_{12}, E_{21},$ *and* E_{22}*, thus* $C^\star(A)$ *is equal to* M_2*. This can also be seen with the graph of* A_G

$G =$ 

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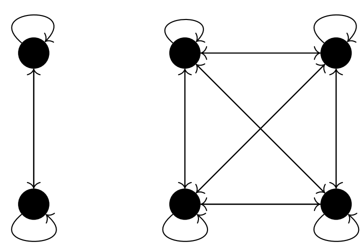
where the dashed edges show the transitive and symmetric closure of G*.*

A_G generates M_n, meaning $C^\star(A_G) = M_n$, if and only if $TC(G \cup \tilde{G}) = K_n$, the complete graph on n vertices. For example,



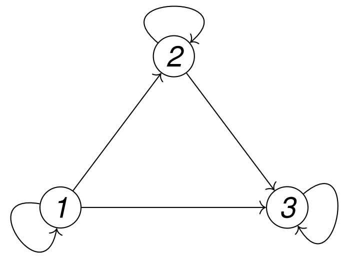
K_2 K_4

Hence, A_G generates M_n exactly when G is weakly connected; that is, $G \cup \tilde{G}$ is connected. Now, we are in a position to study the main topics of this paper.

Definition 2.18. $\mathcal{A} \subseteq M_n$ *is Dirichlet if and only if* $\mathcal{A} + \mathcal{A}^\star = C^\star(\mathcal{A})$*.*

Example 2.19. *Let*

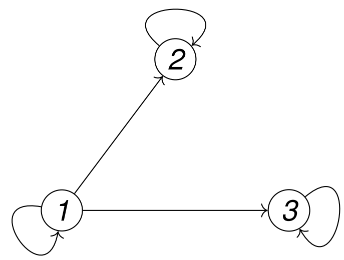
$$A_G = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} =$$



,

A_G *is upper triangular, so from Example 2.16 we know* A_G *is Dirichlet.*

$$A_H = \begin{pmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix} =$$



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A_H *is an algebra and is unital, but is not Dirichlet. Since* $A_H + A_H^\star$ *will be missing the entries in both* E_{23} *and* E_{32}*, and so* $A_H + A_H^\star$ *is smaller than* $C^\star(A_H) = M_3$*.*

Using the graphs can be a quick way to determine if an A_G is Dirichlet.

Definition 2.20. *If* $\mathcal{A}$ *is an algebra and* $S,\ T$ *are subspaces of* $\mathcal{A}$*, then*

$$ST = \text{span}\{st|\ s \in S,\ \text{and}\ t \in T\}.$$

Definition 2.21. *An algebra* $\mathcal{A} \subseteq M_n$ *is semi-Dirichlet if* $\mathcal{A}^\star\mathcal{A} \subseteq \mathcal{A} + \mathcal{A}^\star$*.*

It is immediate that if $\mathcal{A}$ is Dirichlet then $\mathcal{A}$ is semi-Dirichlet, since

$$\mathcal{A}^\star\mathcal{A} \subseteq C^*(\mathcal{A}) = \mathcal{A} + \mathcal{A}^\star.$$

Proposition 2.22. *If* $\mathcal{A}$ *is semi-Dirichlet then,* $C^\star(\mathcal{A}) = \mathcal{A} + \mathcal{A}^\star + \mathcal{A}\mathcal{A}^\star$*. Moreover, if* $\mathcal{A}$ *is unital, then* $C^\star(\mathcal{A}) = \mathcal{A}\mathcal{A}^\star$*.*

Proof. From Corollary 2.14, we know that

$$C^\star(\mathcal{A}) = \mathcal{A} + \mathcal{A}^\star + \mathcal{A}^\star\mathcal{A} + \mathcal{A}\mathcal{A}^\star + \mathcal{A}^\star\mathcal{A}\mathcal{A}^\star + \mathcal{A}\mathcal{A}^\star\mathcal{A} + \cdots,$$

since the closure is not needed as we are working in finite-dimensions. Assume $\mathcal{A}$ is semi-Dirichlet, then $\mathcal{A}^\star\mathcal{A} \subseteq \mathcal{A} + \mathcal{A}^\star$. Every multiplication of two terms or more will have combinations of $\mathcal{A}^\star\mathcal{A}, \mathcal{A}\mathcal{A}, \mathcal{A}^\star\mathcal{A}^\star,$ or $\mathcal{A}\mathcal{A}^\star$. But, $\mathcal{A}^\star\mathcal{A} \subseteq \mathcal{A} + \mathcal{A}^\star$ as $\mathcal{A}$ is semi-Dirichlet, and $\mathcal{A}\mathcal{A} \subseteq \mathcal{A}$ as

$\mathcal{A}$ is closed under multiplication, and $\mathcal{A}^\star\mathcal{A}^\star \subseteq \mathcal{A}^\star$ as $\mathcal{A}^\star$ is closed under multiplication. Notice that every multiplication will simplify to $\mathcal{A}, \mathcal{A}^\star$, or $\mathcal{A}\mathcal{A}^\star$. Therefore, if $\mathcal{A}$ is semi-Dirichlet then $C^\star(\mathcal{A}) = \mathcal{A} + \mathcal{A}^\star + \mathcal{A}\mathcal{A}^\star$. If in addition $\mathcal{A}$ is unital, then $\mathcal{A}, \mathcal{A}^\star \subseteq \mathcal{A}\mathcal{A}^\star$, and so C$^\star(\mathcal{A})$ is just $\mathcal{A}\mathcal{A}^\star$.

□

Using A_H from 2.19 we can see an example of an A_G that is not semi-Dirichlet as,

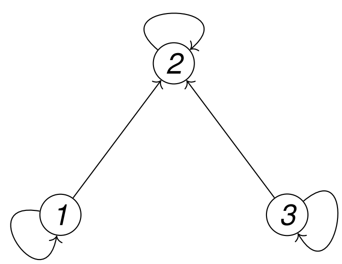
$$A_H + A_H^\star = \begin{pmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} \star & 0 & 0 \\ \star & \star & 0 \\ \star & 0 & \star \end{pmatrix} = \begin{pmatrix} \star & \star & \star \\ \star & \star & 0 \\ \star & 0 & \star \end{pmatrix}, \text{ and}$$

$$A_H^\star A_H = \begin{pmatrix} \star & 0 & 0 \\ \star & \star & 0 \\ \star & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix} = \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix}.$$

$E_{12}^\star E_{13} = E_{21}E_{13} = E_{23} \notin A_H + A_H^\star$, thus A_H is not semi-Dirichlet. In fact, in this example, there are two standard basis vectors in $A_H^\star A_H$ that are not elements of $A_H + A_H^\star$.

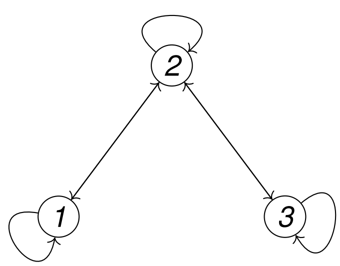
Example 2.23. *Let*

$G =$



, then $A_G = \begin{pmatrix} \star & \star & 0 \\ 0 & \star & 0 \\ 0 & \star & \star \end{pmatrix}$.

Here, $G \cup \tilde{G} =$



, and so A_G *is semi-Dirichlet but is not Dirichlet.*

We can determine whether or not a particular A_G is semi-Dirichlet by examining its graph. As a semi-Dirichlet, A_G will belong to a graph that satisfies a certain condition.

Proposition 2.24. *A_G is semi-Dirichlet if and only if G satisfies the condition that whenever*

$$i \to j \text{ and } i \to k \text{ then either } j \to k \text{ or } k \to j.$$

Proof. ($\Leftarrow$) Suppose G satisfies the above condition. Again, by distributivity, we only need to check the semi-Dirichlet property on the basis vectors. Let E_{ij}, E_{kl} be in A_G. If $i \neq k$, then

$$E_{ij}^\star E_{kl} = E_{ji}E_{kl} = 0 \in A_G + A_G^\star.$$

On the other hand, if $i = k$, then $i \to j$ and $i \to l$ are edges in G. By the assumed property, we have that $j \to l$ or $l \to j$ is an edge in G. In either case, E_{jl} will be in $A_G + A_G^\star$. Then

$$E_{ij}^\star E_{il} = E_{ji}E_{il} = E_{jl} \in A_G + A_G^\star.$$

Therefore, A_G is semi-Dirichlet.

($\Rightarrow$) Suppose A_G is semi-Dirichlet, then $A_G^\star A_G \subseteq A_G + A_G^\star$. Assume $i \to j$ and $i \to k$ are edges in G. So E_{ij} and E_{ik} are in A_G. Thus,

$$E_{jk} = E_{ji}E_{ik} = E_{ij}^\star E_{ik} \in A_G + A_G^\star.$$

Therefore, E_{jk} is in A_G or $A_G^\star$, which means $j \to k$ or $k \to j$ is an edge in G. □

In summary, the correspondence between the properties of the graph G and its corresponding subspace A_G are expressed in the following table:

A_G	G
Algebra	Transitive
Unital	Reflexive
Star-closed	Symmetric or undirected
Star-algebra	Transitive and symmetric
Unital star-algebra	Equivalence relation (union of complete graphs)
Dirichlet	$G \cup \tilde{G}$ is transitive
Semi-Dirichlet	Property in Proposition 2.24
$C^\star(A_G) = M_n$	G is weakly connected

3 Similarity and Examples

The semi-Dirichlet property is invariant under unitary similarity $\mathcal{A} \mapsto U\mathcal{A}U^{-1}$, where U is a unitary matrix. To be unitary U must satisfy either of the following equivalent properties:

(i) $U^\star = U^{-1}$ ($U^\star U = UU^\star = I$).

(ii) U multiplies any orthonormal basis to another orthonormal basis.

Proposition 3.1. *If $\mathcal{A} \subseteq M_n$ is a subalgebra, and U is a unitary then*

$$\mathcal{A} \text{ is semi-Dirichlet if and only if } U\mathcal{A}U^\star = \{UaU^\star |\ a \in \mathcal{A}\} \text{ is semi-Dirichlet.}$$

Proof. If $\mathcal{A}$ is semi-Dirichlet then

$$\begin{aligned}(U\mathcal{A}U^\star)^\star(U\mathcal{A}U^\star) &= (U\mathcal{A}^\star U^\star)(U\mathcal{A}U^\star)\\ &= U\mathcal{A}U^\star U\mathcal{A}U^\star\\ &= U(\mathcal{A}^\star\mathcal{A})U^\star\\ &\subseteq U(\mathcal{A}+\mathcal{A}^*)U^*\\ &= U\mathcal{A}U^* + (U\mathcal{A}U^*)^*.\end{aligned}$$

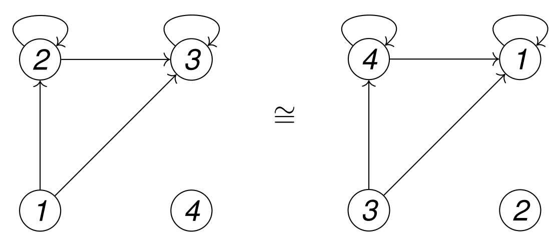
Hence, $U\mathcal{A}U^\star$ is semi-Dirichlet. The other direction follows similarly. □

The simplest situation is when U is a permutation matrix. This corresponds to graph isomorphism. In particular, if G and H are isomorphic graphs via a permutation of the vertices:

$$f : \{1, \cdots, n\} \to \{1, \cdots, n\}.$$

Then, U is the permutation matrix that takes e_i to $e_{f(i)}$ and is such that $U(A_G)U^\star = A_H$.

Example 3.2. *An example of a graph isomorphism.*

$G =$


$= H.$

Then A_G is similar to A_H via the permutation matrix U, where

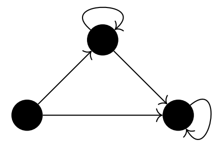
$$A_G = \begin{pmatrix}0&\star&\star&0\\0&\star&\star&0\\0&0&\star&0\\0&0&0&0\end{pmatrix},\ A_H = \begin{pmatrix}\star&0&0&0\\0&0&0&0\\\star&0&0&\star\\\star&0&0&\star\end{pmatrix},\ \text{and } U = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Thus,

$$UA_GU^\star = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&\star&\star&0\\0&\star&\star&0\\0&0&\star&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}\star&0&0&0\\0&0&0&0\\\star&0&0&\star\\\star&0&0&\star\end{pmatrix} = A_H.$$

Proposition 3.1 shows that there are many semi-Dirichlet subalgebras of M_n that don't arise as an A_G, but are unitarily similar to an A_G. For instance:

Example 3.3. *Let*

$G \; =$ 

which is a graph belonging to a semi-Dirichlet A_G as it satisfies the property in propositition 2.24. Then

$$A_G = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{C} \right\}.$$

For the unitary matrix

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } U^{\star} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $\mathcal{A} := U A_G U^{\star}$ is the set of all matrices of the form

$$\begin{aligned} &\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{a-c}{\sqrt{2}} & \frac{b-d}{\sqrt{2}} \\ 0 & \frac{a+c}{\sqrt{2}} & \frac{b+d}{\sqrt{2}} \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{c-a}{2} & \frac{a-c}{2} & \frac{b-d}{\sqrt{2}} \\ \frac{-(a+c)}{2} & \frac{a+c}{2} & \frac{b+d}{\sqrt{2}} \\ 0 & 0 & e \end{pmatrix}. \end{aligned}$$

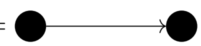
Thus, $\mathcal{A}$ is a semi-Dirichlet operator algebra in M_3, which is clearly not given by a graph.

However, not all $\mathcal{A}$'s are unitarily similar to an A_G. Importantly, unitary similarity preserves many properties of $\mathcal{A}$. To check if an $\mathcal{A}$ is unitarily similar to an A_G, using one of these properties can be useful to narrow down the possible choices of A_G.

Example 3.4. *An example of an operator alegbra that is not similar to an A_G.*

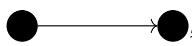
$$\text{Let } \mathcal{A} = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

For a contradiction, assume that $U\mathcal{A}U^\star = A_G$ for some G, then dim(A_G) = dim($\mathcal{A}$) = 1. Then up to another unitary similarity either

$G =$  *, or* $G =$ 

Let $X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, since X satisfies $XX = X$

$$XX = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and $X \neq 0$. If $G =$ *, then every element of $A_G =$ span$\{E_{12}\}$, but then every element of A_G satisfies*

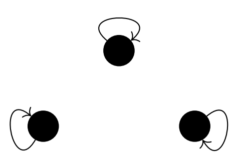
$$X_G X_G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $\mathcal{A}$ cannot be unitarily similar to this A_G.

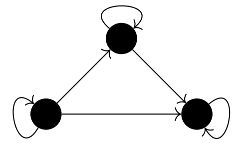
On the other hand, for $G =$ *, $A_G =$ span$\{E_{11}\} = \begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix}$. But $A_G^\star = A_G$, and $\mathcal{A}^\star \neq \mathcal{A}$, and so, $\mathcal{A}$ and A_G cannot be unitarily similar; A_G is self-adjoint and $\mathcal{A}$ is not. Thus, $\mathcal{A}$ is not unitarily similary to any A_G.*

For graphs on 3, 4, and 5 vertices there are 39, 199, and 1049 non-unitarily similar subalgebras of M_n of the form A_G. These subalgebras can be classified into $C^\star$-algebras, Dirichlet algebras, and semi-Dirichlet algebras. It is important to note that these numbers were found by manually creating all directed graphs on 3, 4, and 5 vertices, which satisfy the conditions necessary to be $C^\star$-algebras, Dirichlet algebras, or semi-Dirichlet algebras. On 3 vertices, there are 17 graphs satisfying the semi-Dirichlet condition; on 4 vertices, there are 55; and on 5 vertices, there are 127. The "nicest" ones are listed below. Here "nicest" means the ones that are upper triangular, or are close to being upper triangular.

"Nicest" graphs on 3 vertices are

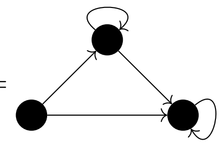
$$C^\star : \begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix} =$$  ,

$$\text{Dirichlet}: \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} =$$



,

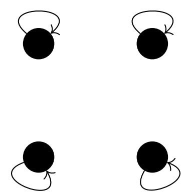
$$\text{semi-Dirichlet}: \begin{pmatrix} 0 & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} =$$



.

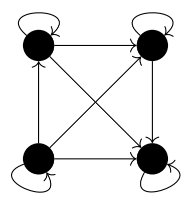
“Nicest” graphs on 4 vertices are

$$C^\star: \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix} =$$



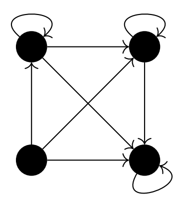
,

$$\text{Dirichlet}: \begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} =$$



,

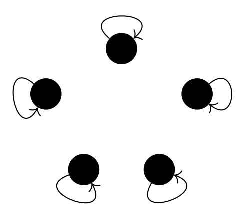
$$\text{semi-Dirichlet}: \begin{pmatrix} 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} =$$



.

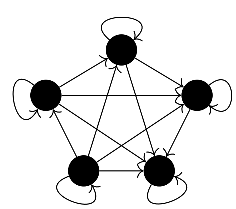
“Nicest” graphs on 5 vertices

$$C^\star: \begin{pmatrix} \star & 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & \star \end{pmatrix} =$$

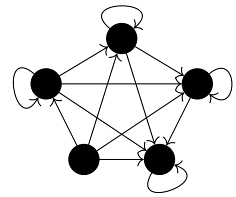


,

$$\text{Dirichlet}: \begin{pmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \star \end{pmatrix} =$$



,

semi-Dirichlet : $\begin{pmatrix} 0 & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \star \end{pmatrix} =$  .

4 Burnside's Theorem

We have seen that there are many subalgebras of M_n that are not unitarily similar to any A_G. However, there is still structure to subalgebras in general. In particular, there is an analogue to Schur's Triangularization Theorem.

Theorem 4.1 (Burnside's Theorem, Th. 1.2.2 **[5]**). *If $\mathcal{A} \subseteq M_n$ is a proper subalgebra, then $\mathcal{A}$ has a proper invariant subspace.*

Note, proper, in this case, means not 0 and not everything. By using Theorem 4.1 we can block triangularize any subalgebra

$$\mathcal{A} \subseteq \left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right).$$

The entries in each diagonal of elements of $\mathcal{A}$ are a subalgebra of M_n.

The procedure is as follows: Let $0 \subsetneq \mathcal{A} \subsetneq M_n$ be a proper subalgebra and $0 \subsetneq V \subsetneq \mathbb{C}^n$ be a proper invariant subspace of $\mathcal{A}$. This means $\mathcal{A}V \subseteq V$. Now choose an orthonormal basis $v_1, \cdots, v_k$ for V and extend to an orthonormal basis $\mathcal{B}$ for $\mathbb{C}^n$. In this basis, every $A \in \mathcal{A}$ has a block matrix decomposition

$$[A]_{\mathcal{B}} = \left(\begin{array}{c|c} B & D \\ \hline 0 & C \end{array}\right).$$

Equivalently $[A]_{\mathcal{B}} = UAU^\star$, where U is the unitary change of basis matrix. Hence, Burnside's Theorem implies that every proper subalgebra is unitarily similar to a block triangular subalgebra. In fact, this is an equivalent formulation. Under the decomposition $\mathbb{C}^n = V \oplus V^\perp$

$$\mathcal{A} \subseteq \left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right).$$

Burnside's Theorem can be applied recursively to block triangularize until all diagonal blocks are either M_n or 0. It is hoped that classifying the semi-Dirichlet property among subalgebras of this form will be more manageable. Perhaps this will happen by replacing graphs with new graphs, whose nodes are subspaces and edges are transformations between them.

The easiest way to get a Burnside's theorem block triangularization is to go the other way.

Example 4.2. *Consider the operator algebra:*

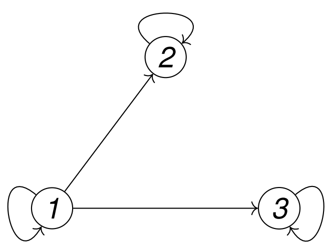
$$A = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix} \; : \; a, b, c, d, e, f, g \in \mathbb{C} \right\}$$

For the unitary $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$*, the algebra*

$$U^*AU = \left\{ \begin{bmatrix} a & \frac{1}{\sqrt{2}}(b+c) & \frac{1}{\sqrt{2}}(b-c) \\ \frac{1}{\sqrt{2}}d & \frac{1}{2}(e+f+g) & \frac{1}{2}(-e+f+g) \\ -\frac{1}{\sqrt{2}}d & \frac{1}{2}(-e-f+g) & \frac{1}{2}(e-f+g) \end{bmatrix} \ : \ a,b,c,d,e,f,g \in \mathbb{C} \right\}$$
$$= \left\{ \begin{bmatrix} a' & b' & c' \\ d' & e'+f'+g' & -e'+f'+g' \\ -d' & -e'-f'+g' & e'-f'+g' \end{bmatrix} \ : \ a',b',c',d',e',f',g' \in \mathbb{C} \right\}$$

We end this paper with an example of proper invariant subspaces.

Example 4.3. *Let*

$G =$  $, \ A_G = \begin{pmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix}.$

The following are proper invariant subspaces for A_G*.*

$$\textit{span}(e_1), \ \ \textit{span}(e_1, e_2), \ \ \textit{span}(e_1, e_3).$$

Note that both 0 and $\mathbb{C}^n$ *are invariant subspaces, but are not proper.*

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