Fermat's Last Theorem and the Golden Mean

Chris Dela Cruz, Leif Fitzsimmons Frey and Tom Luu

Abstract

We attempt to find solutions to the Diophantine equations from Fermat's Last Theorem in the ring $\mathbb{Z}[\tau]$, where τ is the golden mean. We begin with the case when $n = 3$ and create an algorithm to generate solutions to the equation. Out of these solutions, we have found only four to be primitive. Also, we attempt to find solutions under higher powers greater than three but have not found any solutions in such cases. The algorithm and solutions themselves are all provided in the paper.

1 Introduction

In the year 1637, Pierre de Fermat postulated in the Greek text "Arithmetica", that the equation $x^n + y^n = z^n$, where n is an integer greater than 2, will have no positive integer solutions. Supposedly at the time Fermat had a proof for this statement, but unfortunately he passed away before documenting his findings. It was recognized as his "last theorem" because it stood as the only piece of maths from Fermat still yet to be proven. For centuries, mathematicians tried their best to uncover what could've been valid reasoning behind his theorem. Finally, in 1995 Andrew Wiles famously published a proof for Fermat's Last Theorem [4].

This paper attempts to find solutions to the equations in question from Fermat's Last Theorem within specifically $\mathbb{Z}[\tau]$. We first begin solving for $n = 3$, finding out if there exist any solutions. And if there are indeed solutions, we try to delineate which of these solutions are unique. Our team also attempts to do this same thing with higher values such as $n = 4$ and above. This paper will not consider cases for $n = 2$, as that has been completed previously by Marklund-Tweedle in 2021 [3].

We begin in Section 2 introducing the infrastructure required to define $\mathbb{Z}[\tau]$, along with the necessary properties required to find solutions to our equations. First we will explain what τ itself really is, proving basic addition, multiplication, and to show overall that $\mathbb{Z}[\tau]$ is indeed a ring. This allows us to build concepts such as units, and prove the existence of elements that are either irreducible or prime numbers. All of these ideas allow us to build up towards a definition of "primitive" solutions, which is crucial for understanding our problem. Section 3 documents the progress in solving $x^3 + y^3 = z^3$, explaining how we derived our algorithm. Essentially, we took the Diophantine equation in question and expanded it into a system of equations. From this system we began to develop code in order to generate solutions. For $n = 3$, we found four primitive solutions, and have reason to believe that they are the only primitive solutions. For $n > 3$, we were unable to find any solutions at all.

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In section 4, we attempt and fail to find solutions for the Diophantine equations of $n = 4$ and higher. Finally in Section 5, we examine the equations by substitution and expanding the set to $\mathbb{Q}[\tau]$. After some algebraic work we have found non-trivial solutions. We try this by breaking down our previously used strategy and generalizing it for higher powers. All code is provided in their respective sections within our paper.

2 Preliminaries

Before even beginning to find solutions to Fermat's Last Theorem in $\mathbb{Z}[\tau]$, we need to describe $\mathbb{Z}[\tau]$ itself and make sense of its necessary properties. All of the following ring theory can be found in [2].

$$\mathbb{Z}[\tau] := \{a + b\tau : a, b \in \mathbb{Z}\}$$

The roots of $x^2 - x - 1 = 0$ are $\tau = \frac{1+\sqrt{5}}{2}$ and $\tau' = \frac{1-\sqrt{5}}{2}$, where τ is also known as the Golden Ratio.

$\mathbb{Z}[\tau]$ is a ring, which means we can perform addition and multiplication in it. Suppose $x, y \in \mathbb{Z}[\tau]$.

$$x + y = (a_1 + b_1\tau) + (a_2 + b_2\tau) = (a_1 + a_2) + (b_1 + b_2)\tau$$

which holds because $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Z}$.

Before we continue on to multiplication, we need to use a technique that allows us to simplify the multiplication of τ by itself. Since we know $\tau = \frac{1+\sqrt{5}}{2}$ and $\tau' = \frac{1-\sqrt{5}}{2}$, notice that $\frac{1-\sqrt{5}}{2} = 1 - (\frac{1+\sqrt{5}}{2})$ so in other words

$$\tau = 1 - \tau'$$

Also $\tau\tau' = -1$, since $(\frac{1+\sqrt{5}}{2})(\frac{1-\sqrt{5}}{2}) = -1$. Finally if we multiply

$$\tau^2 = \tau\tau = \tau(1 - \tau') = \tau - \tau\tau' = \tau + 1$$

and therefore

$$\tau^2 = \tau + 1$$

We can use the same type of methodology to calculate τ^3 and beyond. And now we can multiply

$$x * y = (a_1 + b_1\tau) * (a_2 + b_2\tau) = a_1a_2 + a_1b_2\tau + b_1a_2\tau + b_1b_2\tau^2$$

Since we derived $\tau^2 = \tau + 1$ we get

$$x * y = a_1a_2 + a_1b_2\tau + b_1a_2\tau + b_1b_2\tau + b_1b_2$$

and we may simplify everything to

$$x * y = (a_1a_2 + b_1b_2) + (a_1b_2 + b_1a_2 + b_1b_2)\tau$$

which holds because $(a_1a_2 + b_1b_2), (a_1b_2 + b_1a_2 + b_1b_2) \in \mathbb{Z}$.

Being a so-called ring of integers $\mathbb{Z}[\tau]$ has a Galois conjugation

$$\sigma : \mathbb{Z}[\tau] \to \mathbb{Z}[\tau] \, ; \, \sigma(m + n\tau) = m + n\tau' \, .$$

σ is a ring isomorphism. It follows immediately that $N : \mathbb{Z}[\tau] \to \mathbb{Z}[\tau]$

$$N(x) = x \cdot \sigma(x)$$

satisfies $N(xy) = N(x)N(y)$. A fast computation shows that

$$N(m + n\tau) = (m + n\tau)(m + n\tau') = m^2 + mn - n^2 \in \mathbb{Z} \, .$$

Next, recall that $a \in \mathbb{Z}[\tau]$ is an unit if there exists some $b \in \mathbb{Z}[\tau]$ such that

$$ab = 1 \, .$$

Now we need to continue understanding preliminary concepts to flesh out the notion of a "primitive" solution inside of $\mathbb{Z}[\tau]$. We must begin with units.

Fact 2.1. *An element $a \in \mathbb{Z}[\tau]$ is an unit if and only if*

$$N(a) = \pm 1 \, .$$

Units allows us to describe the ideas of irreducible and prime elements of $\mathbb{Z}[\tau]$ to build a Unique Factorization Domain (UFD).

2.1 UFD

Let us recall the following definitions:

Definition 2.2. *An element $a \in \mathbb{Z}[\tau]$ is called **irreducible** if whenever $a = bc$ we have b is an unit or c is an unit.*

*An element $p \in \mathbb{Z}[\tau]$ is called **prime** if whenever $p|bc$ we have $p|b$ or $p|c$.*

The following is a well-known fact in any integral domain.

Fact 2.3. *If p is prime, then p is irreducible.*

Definition 2.4. *A* **Unique Factorization Domain (UFD)** *Z is an integral domain such that any element $a \in Z$ can be written in as a product of irreducibles*

$$a = q_1 q_2 \ldots q_m \, .$$

Moreover, this writing is unique in the sense that if

$$a = p_1 p_2 \ldots p_k$$

is a product of irreducible elements then $k = m$ and there exists a bijection $\phi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ and units $u_1, \ldots, u_m \in Z$ such that

$$p_j = u_j q_{\phi(j)} \, .$$

Theorem 2.5. *[3, Thm. 2.3] $\mathbb{Z}[\tau]$ is an Euclidean domain. In particular,$\mathbb{Z}[\tau]$ is a UFD and every irreducible is a prime element.* □

Now that $\mathbb{Z}[\tau]$ is properly understood as a UFD, we can now go ahead and introduce the idea of a gcd (**greatest common divisor**).

Since $\mathbb{Z}[\tau]$ is an UFD, every two elements $a, b \in \mathbb{Z}[\tau]$ with $(a, b) \neq (0, 0)$ have a $\gcd(a, b)$. Formally, a greatest common divisor of a, b is any element $d \in \mathbb{Z}[\tau]$ which satisfies

- $d|a, d|b$
- If $e|a, e|b$ then $e|d$.

Recall here that if d is a gcd of a, b, then d' is a gcd of a, b if and only if there exists a unit u such that $d' = ud$.

We will simply denote by $\gcd(a, b)$ any gcd of a, b. Moreover, since $\mathbb{Z}[\tau]$ is an Euclidean domain, $\gcd(a, b)$ can be calculated via the Euclidean algorithm.

2.2 Diophantine equations

An equation is called Diophantine if there are two or more unknowns with integer coefficients. Traditionally, these equations are mostly considered for solutions where the unknowns are all integers. One of the most famous Diophantine equations is Fermat's Last Theorem.

Fermat's Last Theorem states that no three integers x, y, and z can satisfy the exponential Diophantine equation

$$x^n + y^n = z^n$$

if $n > 2$, $n \in \mathbb{Z}$.

Famously proven by Andrew Wiles in 1995 [4], our team decided to find out if there existed solutions to Fermat's Last Theorem where instead x, y, and z are elements in the $\mathbb{Z}[\tau]$. Formally we can describe our problem as the following.

Question 2.6. *Are there solutions to the exponential Diophantine equation*

$$x^n + y^n = z^n$$

if $n > 2$ when $x, y, z \in \mathbb{Z}[\tau]$?

We initially suspected that there would indeed be solutions when $n = 3$, so we decided to focused our energy onto $n = 3$ specifically before moving forward to higher values.

Every scalar multiple of a solution is itself a solution. Trivially, this means that there are an infinite number of solutions, but this does not interest us since these solutions are effectively the same. In order to filter these solutions out, we define some solutions as primitive.

Definition 2.7. *A triple (x, y, z) satisfying $x^n + y^n = z^n$ is called a **primitive solution** if*

$$\gcd(x, y, z) = 1\,.$$

Operating under this definition, any scalar multiple of a primitive solution is nonprimitive; therefore, no set of primitive solutions will contain solutions which are scalar multiples of each other. This removes the ”duplicate” solutions, and should make understanding the underlying structure easier.

Any solution(x, y, z) can be written in the form

$$\begin{cases} x &= dx' \\ y &= dy' \\ z &= dz' \end{cases}$$

where $d = \gcd(x, y, z)$ and (x', y', z') is a primitive solution. Moreover, this writing is unique up to multiplication by units (or equivalent choice of gcd).

3 Solutions to $n = 3$

We begin by breaking down our previously stated problem and setting $n = 3$.

Question 3.1. *Are there any solutions to the Diophantine equation $x^3 + y^3 = z^3$ such that $x, y, z \in \mathbb{Z}[\tau]$?*

We can then break down each x, y, z into their respective integer and tau components, where we attempt to find solutions to

$$(a + b\tau)^3 + (c + d\tau)^3 = (e + f\tau)^3$$

where $a, b, c, d, e, f \in \mathbb{Z}$.

To solve this equation, we can expand each pair. Using binomial expansion we can see that

$$(a + b\tau)^3 = a^3 + 3a^2b\tau + 3ab^2\tau^2 + b^3\tau^3$$

$$(c + d\tau)^3 = c^3 + 3c^2d\tau + 3cd^2\tau^2 + d^3\tau^3$$

$$(e + f\tau)^3 = e^3 + 3e^2f\tau + 3ef^2\tau^2 + df^3\tau^3$$

The most difficult part of this expansion is the τ^2 and τ^3, but it can be handled by reducing it down into

$$\tau^2 = \tau + 1$$

$$\tau^3 = 2\tau + 1$$

Substituting those formula into the equations allow the expansions to become

$$(a + b\tau)^3 = a^3 + 3a^2b\tau + 3ab^2\tau + 3ab^2 + 2b^3\tau + b^3$$

$$(c + d\tau)^3 = c^3 + 3c^2d\tau + 3cd^2\tau + 3cd^2 + 2d^3\tau + d^3$$

$$(e + f\tau)^3 = e^3 + 3e^2 f\tau + 3ef^2\tau + 3ef^2 + 2f^3\tau + f^3$$

So altogether we can then move the expanded terms of $(e + f\tau)^3$ to the left hand side, to get

$$(a^3 + 3a^2b\tau + 3ab^2\tau + 3ab^2 + 2b^3\tau + b^3) + (c^3 + 3c^2d\tau + 3cd^2\tau + 3cd^2 + 2d^3\tau + d^3) -$$

$$(e^3 + 3e^2f\tau + 3ef^2\tau + 3ef^2 + 2f^3\tau + f^3) = 0$$

Notice that we can now factor out τ out of many of the terms, thus leading us to once again break down this equation into an integer and tau components, giving us the system of equations

$$\begin{cases} a^3 + 3ab^2 + b^3 + c^3 + 3cd^2 + d^3 - e^3 - 3ef^2 - f^3 = 0 \\ 3a^2b + 3ab^2 + 2b^3 + 3c^2d + 3cd^2 + 2d^3 - 3e^2f - 3ef^2 - 2f^3 = 0 \end{cases}$$

To find solutions we can then use a computer to calculate for which instances of $a, b, c, d, e, f \in \mathbb{Z}$ does the system of equation above hold. We can make a simple script that does this in python. First let's make a function defining the two equations we aim to solve for.

```
# Calcuates system of equations when choosing each term
def calc_eq(a, b, c, d, e, f):
    eq1 = (a**3 + 3*a*b**2 + b**3 + c**3 + 3*c*d**2 + d**3
           - e**3 - 3*e*f**2 - f**3)
    eq2 = (3*a**2*b + 3*a*b**2 + 2*b**3 + 3*c**2*d + 3*c*d**2 + 2*d**3
           - 3*e**2*f - 3*e*f**2 - 2*f**3)
    return eq1, eq2
```

In the meantime we can also conveniently at this stage calculate the respective norms of each pair. Defining a function for these norms can be done by following the computation outlined above.

```
# Calculates norms of each triple
def calc_n(a, b, c, d, e, f):
    norms1 = a**2 + a*b - b**2
    norms2 = c**2 + c*d - d**2
    norms3 = e**2 + e*f - f**2
    return norms1, norms2, norms3
```

We can then test every possible value of a, b, c, d, e, f in a given range. To do so, we can use six nested for loops to iterate through values of a through f. This is tedious but straightforward:

```
import math

sol = []                    # List where solutions are stored
max = 20                    # Search space

for a in range(1, max):
    for b in range(-max, max):
```

```
for c in range(1, max):
    for d in range(-max, max):
        for e in range(-max, max):
            for f in range(-max, max):
                eq1, eq2 = calc_eq(a, b, c, d, e, f)
                if eq1 == 0 and eq2 == 0:
                    n1, n2, n3 = calc_n(a, b, c, d, e, f)
                    sol.append([a, b, c, d, e, f, n1, n2, n3])
```

Alternatively, we can store possible solutions as an array of length 6, and use a function to "increment" the array by 1, treating it as a 6-digit number of arbitrary base.

We can increase the max value to increase the search space, although this naturally increases computational resources required. Our team has found 120 solutions. The first six are as follows and the remainder can be found in the Appendix.

$$(1-14\tau)^3+(8-13\tau)^3=(6-18\tau)^3 \qquad (1+15\tau)^3+(8+21\tau)^3=(6+24\tau)^3$$
$$(2-9\tau)^3+(7-9\tau)^3=(6-12\tau)^3 \qquad (2+11\tau)^3+(7+16\tau)^3=(6+18\tau)^3$$
$$(3-4\tau)^3+(6-5\tau)^3=(6-6\tau)^3 \qquad (3+7\tau)^3+(6+11\tau)^3=(6+12\tau)^3$$

A quick observation of these solutions show that many of these solutions tend to be multiples of the other. For example, $(2-9\tau)^3+(7-9\tau)^3=(6-12\tau)^3$ is the same thing as $(4-18\tau)^3+(14-18\tau)^3=(12-24\tau)^3$ except multiplied by two for each term. Hence it's easy to see that several of these solutions are not primitive.

In that case it might be worth trying to find out of all these solutions which ones should be primitive. There are potentially several ways of discovering them, but we have decided to attempt to solve

$$x^3+y^3+z^3=0$$

where $x, y, z \in \mathbb{Z}[\tau]$. When creating the system of equations previously we subtracted the z term, but rewriting it in this form will make this method simpler.

Now if we divide both sides by z^3, we end up with

$$\frac{x^3}{z^3}+\frac{y^3}{z^3}+1=0$$

In other words, for any solution we have for (x, y, z) there is immediately another triple $(\frac{x}{z}, \frac{y}{z}, 1)$. Though we must assume that the z component is the one that has the highest absolute value norm.

In formulating this as an algorithm, we can compare each solution to each other. Divide a triple by it's z term, and check if it's a unique value. Any new values of $(\frac{x}{z}, \frac{y}{z})$ unlike the others can be deemed a primitive solution. Translated into python, we can create a script alike the following below.

```
T = (1 + math.sqrt(5)) / 2  # Tau value

# 'sol'  is the list of previously generated solutions
# 'prim' is the list of primitive solutions
prim = []

# For each entry in the list, calculate (x/z, y/z)
for i in sol:
    x_z = (i[0]+ i[1]*T)/(i[4]+ i[5]*T)
    y_z = (i[2]+ i[3]*T)/(i[4]+ i[5]*T)

    for j in sol:

        # Add any unique (x/z, y/z) entries to our primitives list.
        if (j != [x_z,y_z]) and (j != [y_z,x_z]):
            prim.append([x_z, y_z])
```

Interestingly when running this program we observe that we end up with only two primitive solutions

$$(0.9363389981249823, 0.5636610018750172) = \left(\frac{4+\tau}{6}, \frac{5-\tau}{6}\right)$$

$$(0.5636610018750176, 0.9363389981249824) = \left(\frac{5-\tau}{6}, \frac{4+\tau}{6}\right)$$

Because $\tau = \frac{1+\sqrt{5}}{2}$, we can deduce that these two values come from the solution $(4+1\tau)^3 + (5-1\tau)^3 = (6+0\tau)^3$ and conversely $(5-1\tau)^3 + (4+1\tau)^3 = (6+0\tau)^3$. And including the non-trivial solution and it's flipped version we can establish a conjecture.

Conjecture 3.2. *The equation $x^3 + y^3 = z^3$ where $x, y, z \in \mathbb{Z}[\tau]$ has four primitive solutions.*

$$(4+\tau)^3 + (5-\tau)^3 = 6^3$$

$$(5-\tau)^3 + (4+\tau)^3 = 6^3$$

$$1^3 + 0^3 = 1^3$$

$$0^3 + 1^3 = 1^3$$

4 Solutions to $n > 3$

The same algorithm in the previous section is used, but instead of a specific formula to cube an integer + tau pair, we use the more general formula to multiply two such pairs together:

$$(a+b\tau)(c+d\tau) = ac + ad\tau + bc\tau + bd\tau^2$$

$$= ac + ad\tau + bc\tau + bd(\tau + 1)$$
$$= ac + ad\tau + bc\tau + bd\tau + bd$$
$$= ac + bd + (ad + bc + bd)\tau$$

Any pair can then be multiplied by itself an arbitrary number of times to achieve any arbitrary *n*-value. The rest of the search algorithm proceeds as normal. We were unable to find any non-trivial solutions for $n > 3$.

Theorem 4.1. *There are no non-trivial solutions in* $\mathbb{Z}[\tau]$ *for* $(a + b\tau)^n + (c + d\tau)^n = (e + f\tau)^n$ *for* $4 \leqslant n \leqslant 7$ *and* $-10 \leqslant a, b, c, d, e, f \leqslant 10$ *or* $0 \leqslant a, b, c, d, e, f \leqslant 20$.

Proof. The proof is done by running Algorithm 3 on page 6. □

Conjecture 4.2. *For* $n \geqslant 4$*, there are no non-trivial solutions in* $\mathbb{Z}[\tau]$ *for* $x^n + y^n = z^n$.

5 Reducing the equation in $\mathbb{Z}[\tau]$

We examined the algebraic structure of these Diophantine equations to see if we could prove some of our conjectures. Of course, this led to more conjectures.

If we start with $x^3 + y^3 = z^3$, denoting $u = \frac{x}{z}$, $v = \frac{y}{z}$, we have

$$u^3 + v^3 = 1$$
$$(u + v)(u^2 - uv + v^2) = 1$$
$$(u + v)((u + v)^2 - 3uv) = 1$$

Let $r = u + v$, $s = u \cdot v$, we have:

$$r^3 - 3rs = 1$$

with $r, s \in \mathbb{Q}[\tau]$

Through testing, the only non-trivial found solution is $u = \frac{5}{6} - \frac{1}{6}\tau$, $v = \frac{2}{3} + \frac{1}{6}\tau$, $1^3 + 0^3 = 1^3$, $0^3 + 1^3 = 1^3$ which means $r = \frac{3}{2}$ and $s = \frac{19}{36}$. So all solutions of $x^3 + y^3 = z^3$ lead to rational solutions of $r^3 - 3rs = 1$. But this isn't so helpful since the latter equation has an infinite number of rational solutions, e.g. let $r \in \mathbb{Q}$ and solve for s.

This leads us to believe that r and s are always rational.

Let's examine this by letting

$$u = \frac{m + n\tau}{p}$$
$$v = \frac{k + l\tau}{q}$$

for $m, n, k, l, p, q \in \mathbb{Z}$ with $u^3 + v^3 = 1$. Then

$$\begin{aligned} s &= uv \\ &= \frac{mk + (ml + nk)\tau + kl\tau^2}{pq} \\ &= \frac{mk + kl + (ml + nk + kl)\tau}{pq} \end{aligned}$$

Conjecture 5.1. *Let $u, v \in \mathbb{Z}[\tau]$ are so that $u^3 + v^3 = 1$. Then $s = uv$ satisfies either $s = 0$ or $19 \mid mk + kl$ and $ml + nk + kl = 0$,*

We found 4 non-trivial solutions for the triple

$$(x, y, z) = (4 + \tau, 5 - \tau, 6), (5 - \tau, 4 + \tau, 6), (1, 0, 1), (0, 1, 1)\,,$$

which translates into

$$(u, v) = \left(\frac{2}{3} + \frac{\tau}{6}, \frac{5}{6} - \frac{\tau}{6}\right), \left(\frac{5}{6} - \frac{\tau}{6}, \frac{2}{3} + \frac{\tau}{6}\right), (1, 0), (0, 1)\,.$$

It is noticeable that

$$\begin{aligned} \sigma\left(\frac{2}{3} + \frac{\tau}{6}\right) &= \frac{2}{3} + \frac{\tau'}{6} \\ &= \frac{2}{3} + \frac{1 - \tau}{6} \\ &= \frac{2}{3} + \frac{1}{6} - \frac{\tau}{6} \\ &= \frac{5}{6} - \frac{\tau}{6} \end{aligned}$$

and conversely, $\sigma\left(\frac{5}{6} - \frac{\tau}{6}\right) = \frac{2}{3} + \frac{\tau}{6}$

Conjecture 5.2. *For all non-trivial solutions of $u^3 + v^3 = 1$ with $u, v \in \mathbb{Q}[\tau]$ we have $v = \sigma(u)$.*

6 Conclusion

The brute-force search algorithms found no solutions for $n \geqslant 4$ and created a long list of solutions for $n = 3$, some of which are noticeably proportional, like $(13+5\tau)^3 + (14+13\tau)^3 = (18+12\tau)^3$ and $(26 + 10\tau)^3 + (28 + 26\tau)^3 = (36 + 24\tau)^3$, but some are less noticeable, for example, we consider $(14 + \tau)^3 + (13 + 8\tau)^3 = (18 + 6\tau)^3$ and $(13 + 5\tau)^3 + (14 + 13\tau)^3 = (18 + 12\tau)^3$

$$\frac{14 + \tau}{13 + 5\tau} = \frac{14 \cdot 13 - 1 \cdot 5 + 14 \cdot 5 + (13 - 14 \cdot 5)\tau}{13^2 + 13 \cdot 5 - 5^2} = \frac{13 - 3\tau}{11}$$

$$\frac{13 + 8\tau}{14 + 13\tau} = \frac{13 \cdot 14 - 8 \cdot 13 + 13^2 + (14 \cdot 8 - 13^2)\tau}{14^2 + 14 \cdot 13 - 13^2} = \frac{13 - 3\tau}{11}$$

$$\frac{18+6\tau}{18+12\tau}=\frac{3+\tau}{3+2\tau}=\frac{3^2-2+3\cdot 2+(3-6)\tau}{3^2+3\cdot 2-2^2}=\frac{13-3\tau}{11}$$

Therefore, the 2 equations are proportional by $\frac{13-3\tau}{11}$.

In order to find primitive solutions in the list, we set one element as the anchor, then compare the others.

$$x^3+y^3=z^3$$

$$\frac{x^3}{z^3}+\frac{y^3}{z^3}=1$$

Therefore, any triple (x,y,z) can be transformed into $(\frac{x}{z},\frac{y}{z},1)$, if they are proportional, only one is retained. From the original list of 120 solutions, we only found 4 primitive solutions:

$$(4+\tau)^3+(5-\tau)^3=6^3$$

$$(5-\tau)^3+(4+\tau)^3=6^3$$

$$1^3+0^3=1^3$$

$$0^3+1^3=1^3$$

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Appendix: Solutions for $n = 3$

$$(1-14\tau)^3+(8-13\tau)^3=(6-18\tau)^3 \qquad (1+15\tau)^3+(8+21\tau)^3=(6+24\tau)^3$$
$$(2-9\tau)^3+(7-9\tau)^3=(6-12\tau)^3 \qquad (2+11\tau)^3+(7+16\tau)^3=(6+18\tau)^3$$
$$(3-4\tau)^3+(6-5\tau)^3=(6-6\tau)^3 \qquad (3+7\tau)^3+(6+11\tau)^3=(6+12\tau)^3$$
$$(4-18\tau)^3+(14-18\tau)^3=(12-24\tau)^3 \qquad (4+1\tau)^3+(5-1\tau)^3=(6+0\tau)^3$$
$$(4+3\tau)^3+(5+6\tau)^3=(6+6\tau)^3 \qquad (5-13\tau)^3+(13-14\tau)^3=(12-18\tau)^3$$
$$(5-1\tau)^3+(4+1\tau)^3=(6+0\tau)^3 \qquad (5+6\tau)^3+(4+3\tau)^3=(6+6\tau)^3$$
$$(6-8\tau)^3+(12-10\tau)^3=(12-12\tau)^3 \qquad (6-5\tau)^3+(3-4\tau)^3=(6-6\tau)^3$$
$$(6+11\tau)^3+(3+7\tau)^3=(6+12\tau)^3 \qquad (6+14\tau)^3+(12+22\tau)^3=(12+24\tau)^3$$
$$(7-9\tau)^3+(2-9\tau)^3=(6-12\tau)^3 \qquad (7-3\tau)^3+(11-6\tau)^3=(12-6\tau)^3$$
$$(7+10\tau)^3+(11+17\tau)^3=(12+18\tau)^3 \qquad (7+16\tau)^3+(2+11\tau)^3=(6+18\tau)^3$$
$$(8-17\tau)^3+(19-19\tau)^3=(18-24\tau)^3 \qquad (8-13\tau)^3+(1-14\tau)^3=(6-18\tau)^3$$
$$(8+2\tau)^3+(10-2\tau)^3=(12+0\tau)^3 \qquad (8+6\tau)^3+(10+12\tau)^3=(12+12\tau)^3$$
$$(8+21\tau)^3+(1+15\tau)^3=(6+24\tau)^3 \qquad (9-12\tau)^3+(18-15\tau)^3=(18-18\tau)^3$$
$$(9+2\tau)^3+(9+7\tau)^3=(12+6\tau)^3 \qquad (9+7\tau)^3+(9+2\tau)^3=(12+6\tau)^3$$
$$(10-7\tau)^3+(17-11\tau)^3=(18-12\tau)^3 \qquad (10-2\tau)^3+(8+2\tau)^3=(12+0\tau)^3$$
$$(10+12\tau)^3+(8+6\tau)^3=(12+12\tau)^3 \qquad (11-6\tau)^3+(7-3\tau)^3=(12-6\tau)^3$$
$$(11-2\tau)^3+(16-7\tau)^3=(18-6\tau)^3 \qquad (11+13\tau)^3+(16+23\tau)^3=(18+24\tau)^3$$
$$(11+17\tau)^3+(7+10\tau)^3=(12+18\tau)^3 \qquad (12-16\tau)^3+(24-20\tau)^3=(24-24\tau)^3$$
$$(12-10\tau)^3+(6-8\tau)^3=(12-12\tau)^3 \qquad (12+3\tau)^3+(15-3\tau)^3=(18+0\tau)^3$$
$$(12+9\tau)^3+(15+18\tau)^3=(18+18\tau)^3 \qquad (12+22\tau)^3+(6+14\tau)^3=(12+24\tau)^3$$
$$(13-14\tau)^3+(5-13\tau)^3=(12-18\tau)^3 \qquad (13-11\tau)^3+(23-16\tau)^3=(24-18\tau)^3$$
$$(13+5\tau)^3+(14+13\tau)^3=(18+12\tau)^3 \qquad (13+8\tau)^3+(14+1\tau)^3=(18+6\tau)^3$$
$$(14-18\tau)^3+(4-18\tau)^3=(12-24\tau)^3 \qquad (14-6\tau)^3+(22-12\tau)^3=(24-12\tau)^3$$
$$(14+1\tau)^3+(13+8\tau)^3=(18+6\tau)^3 \qquad (14+13\tau)^3+(13+5\tau)^3=(18+12\tau)^3$$
$$(15-3\tau)^3+(12+3\tau)^3=(18+0\tau)^3 \qquad (15-1\tau)^3+(21-8\tau)^3=(24-6\tau)^3$$
$$(15+18\tau)^3+(12+9\tau)^3=(18+18\tau)^3 \qquad (16-7\tau)^3+(11-2\tau)^3=(18-6\tau)^3$$
$$(16+4\tau)^3+(20-4\tau)^3=(24+0\tau)^3 \qquad (16+12\tau)^3+(20+24\tau)^3=(24+24\tau)^3$$
$$(16+23\tau)^3+(11+13\tau)^3=(18+24\tau)^3 \qquad (17-11\tau)^3+(10-7\tau)^3=(18-12\tau)^3$$
$$(17+8\tau)^3+(19+19\tau)^3=(24+18\tau)^3 \qquad (17+9\tau)^3+(19+0\tau)^3=(24+6\tau)^3$$
$$(18-15\tau)^3+(9-12\tau)^3=(18-18\tau)^3 \qquad (18+4\tau)^3+(18+14\tau)^3=(24+12\tau)^3$$
$$(18+14\tau)^3+(18+4\tau)^3=(24+12\tau)^3 \qquad (19-19\tau)^3+(8-17\tau)^3=(18-24\tau)^3$$
$$(19+0\tau)^3+(17+9\tau)^3=(24+6\tau)^3 \qquad (19+19\tau)^3+(17+8\tau)^3=(24+18\tau)^3$$

$$(20-4\tau)^3+(16+4\tau)^3=(24+0\tau)^3$$
$$(21-8\tau)^3+(15-1\tau)^3=(24-6\tau)^3$$
$$(23-16\tau)^3+(13-11\tau)^3=(24-18\tau)^3$$
$$(25-27\tau)^3+(47-36\tau)^3=(48-42\tau)^3$$
$$(25+11\tau)^3+(29-2\tau)^3=(36+6\tau)^3$$
$$(26-22\tau)^3+(46-32\tau)^3=(48-36\tau)^3$$
$$(26+10\tau)^3+(28+26\tau)^3=(36+24\tau)^3$$
$$(27-17\tau)^3+(45-28\tau)^3=(48-30\tau)^3$$
$$(27+6\tau)^3+(27+21\tau)^3=(36+18\tau)^3$$
$$(27+25\tau)^3+(36+47\tau)^3=(42+48\tau)^3$$
$$(28+2\tau)^3+(26+16\tau)^3=(36+12\tau)^3$$
$$(28+21\tau)^3+(35+42\tau)^3=(42+42\tau)^3$$
$$(29-7\tau)^3+(43-20\tau)^3=(48-18\tau)^3$$
$$(29+17\tau)^3+(34+37\tau)^3=(42+36\tau)^3$$
$$(30+13\tau)^3+(33+32\tau)^3=(42+30\tau)^3$$
$$(31+3\tau)^3+(41-12\tau)^3=(48-6\tau)^3$$
$$(31+22\tau)^3+(32+5\tau)^3=(42+18\tau)^3$$
$$(32+8\tau)^3+(40-8\tau)^3=(48+0\tau)^3$$
$$(32+27\tau)^3+(31+9\tau)^3=(42+24\tau)^3$$
$$(33+13\tau)^3+(39-4\tau)^3=(48+6\tau)^3$$
$$(33+32\tau)^3+(30+13\tau)^3=(42+30\tau)^3$$
$$(34+16\tau)^3+(38+38\tau)^3=(48+36\tau)^3$$
$$(34+37\tau)^3+(29+17\tau)^3=(42+36\tau)^3$$
$$(34+18\tau)^3+(38+0\tau)^3=(48+12\tau)^3$$
$$(35+23\tau)^3+(37+4\tau)^3=(48+18\tau)^3$$
$$(36+28\tau)^3+(36+8\tau)^3=(48+24\tau)^3$$
$$(37+33\tau)^3+(35+12\tau)^3=(48+30\tau)^3$$
$$(38+38\tau)^3+(34+16\tau)^3=(48+36\tau)^3$$

$$(20+24\tau)^3+(16+12\tau)^3=(24+24\tau)^3$$
$$(22-12\tau)^3+(14-6\tau)^3=(24-12\tau)^3$$
$$(24-20\tau)^3+(12-16\tau)^3=(24-24\tau)^3$$
$$(25-8\tau)^3+(38-19\tau)^3=(42-18\tau)^3$$
$$(25+14\tau)^3+(29+31\tau)^3=(36+30\tau)^3$$
$$(26-3\tau)^3+(37-15\tau)^3=(42-12\tau)^3$$
$$(26+16\tau)^3+(28+2\tau)^3=(36+12\tau)^3$$
$$(27+2\tau)^3+(36-11\tau)^3=(42-6\tau)^3$$
$$(27+21\tau)^3+(27+6\tau)^3=(36+18\tau)^3$$
$$(28-12\tau)^3+(44-24\tau)^3=(48-24\tau)^3$$
$$(28+7\tau)^3+(35-7\tau)^3=(42+0\tau)^3$$
$$(28+26\tau)^3+(26+10\tau)^3=(36+24\tau)^3$$
$$(29+12\tau)^3+(34-3\tau)^3=(42+6\tau)^3$$
$$(30-2\tau)^3+(42-16\tau)^3=(48-12\tau)^3$$
$$(30+17\tau)^3+(33+1\tau)^3=(42+12\tau)^3$$
$$(31+9\tau)^3+(32+27\tau)^3=(42+24\tau)^3$$
$$(32+5\tau)^3+(31+22\tau)^3=(42+18\tau)^3$$
$$(32+24\tau)^3+(40+48\tau)^3=(48+48\tau)^3$$
$$(33+1\tau)^3+(30+17\tau)^3=(42+12\tau)^3$$
$$(33+20\tau)^3+(39+43\tau)^3=(48+42\tau)^3$$
$$(34-3\tau)^3+(29+12\tau)^3=(42+6\tau)^3$$
$$(34+18\tau)^3+(38+0\tau)^3=(48+12\tau)^3$$
$$(34+16\tau)^3+(38+38\tau)^3=(48+36\tau)^3$$
$$(35+12\tau)^3+(37+33\tau)^3=(48+30\tau)^3$$
$$(36+8\tau)^3+(36+28\tau)^3=(48+24\tau)^3$$
$$(37+4\tau)^3+(35+23\tau)^3=(48+18\tau)^3$$
$$(38+0\tau)^3+(34+18\tau)^3=(48+12\tau)^3$$
$$(38-38\tau)^3+(70-52\tau)^3=(72-60\tau)^3$$