Van der Pol Oscillator – Analysis of a Non-conservative System

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Goals of the Thesis

- Understand the history of the Van der Pol Oscillator and relaxation oscillations.

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- Analyse the Van der Pol oscillator using both an intuitive approach and an analytical approach to the first order averaging method.
- Discover and apply methods from perturbation theory to the Van der Pol oscillator.

Note: This presentation will cover the material of the thesis up to the intuitive approach of understanding Van der Pol oscillator

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Self-Sustaining Oscillating Systems

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- Examples include a mass on a spring or an LC circuit.

LC circuit

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LC circuit

- Circuit containing an inductor and a capacitor.
- Describes a simple harmonic oscillator.

Relaxation Oscillators

- Oscillating systems emerging from nonlinear restoring forces.[1]
- Examples include circuits with triodes and RLC series circuits.
- Term coined by Van der Pol.[1]

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- Derived the Van der Pol equation while studying triode circuits. [1]

Van der Pol Equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{1.1}$$

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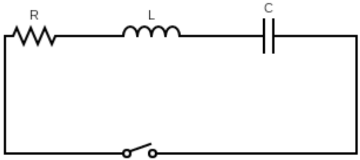

Figure: Example of an RLC series circuit

RLC Series Circuit

- We will supply the RLC circuit with a voltage of v(t) volts (V).

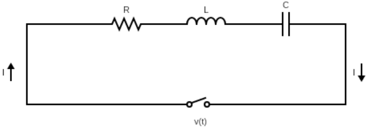

Figure: RLC Series circuit supplied with v(t) volts and current travelling clockwise.

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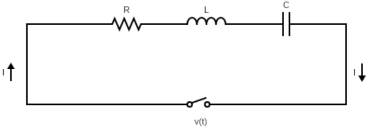

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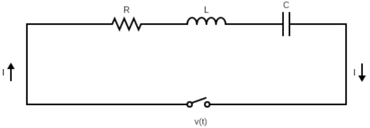

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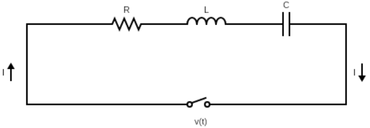

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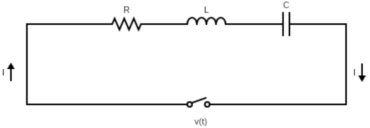
- The intensity over electrical current $I = I(t)$ is defined as

$$I = \frac{dQ}{dt}. \qquad (2.1)$$



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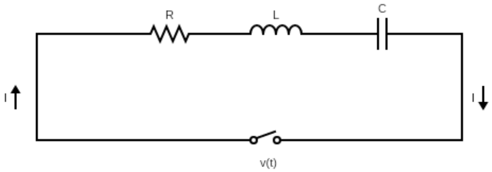
Kirchhoff's Law



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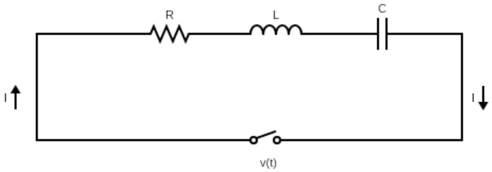
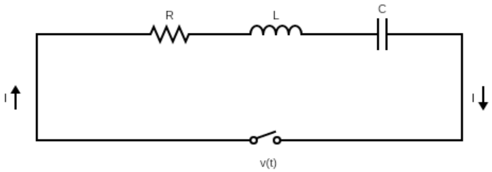


Kirchhoff's voltage law

The sum of all voltage drops around a closed loop equals to zero.

Kirchhoff's Law



Applying Kirchhoff's voltage law to our circuit, we find

$$V_R + V_L + V_C = V(t). \tag{2.2}$$

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- The voltage drop on the capacitor is:

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or equivalently

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = v(t). \tag{2.8}$$

(2.8) is the ODE describing the charge over the capacitor.

ODE for Intensity of Electrical Current

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which is the ODE describing the intensity of the electric current in the circuit.

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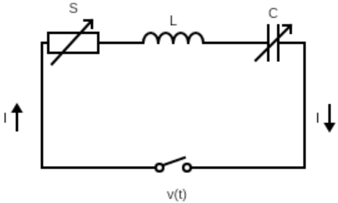

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We have

$$I = \frac{dQ}{dt} \tag{2.11}$$

and

$$V_c = \frac{1}{C(t)} Q \tag{2.12}$$

thus

$$I = \frac{d(V_C \cdot C)}{dt} = (V_C \cdot C)' \tag{2.13}$$

Deriving the Lieńard Equation

To derive the Lieńard equation, we will apply Kirchhoff's Law of voltage in a closed loop to our new circuit

$$V_S + V_L + V_C = v(t) \tag{2.14}$$

where

$$V_s = F(I) \tag{2.15}$$

is a nonlinear function of I, which we choose to be differentiable. Combining (2.4), (2.14) and (2.15), we get

$$F(I) + L\frac{dI}{dt} + V_C = v(t). \tag{2.16}$$

Deriving the Lieńard Equation

Multiplying both sides of the equation by C, in order to group V_C and C, we get

$$CF(I) + CLI' + C \cdot V_C = Cv(t).$$

Taking the derivative of both sides with respect to t gives us:

$$C'F(I) + CF'(I)I' + C'LI' + CLI'' + (C \cdot V_C)' = (Cv(t))'$$

Applying (2.13), we get:

$$\begin{aligned} C'F(I) + CF'(I)I' + C'LI' + CLI'' + I &= (Cv(t))' \\ CLI'' + (CF'(I) + C'L)I' + (C'F(I) + I - (Cv(t'')) &= 0 \end{aligned}$$

Putting the latter equation in standard form, we get:

$$I'' + \frac{1}{CL}(CF'(I) + C'L)I' + \frac{1}{CL}(C'F(I) + I - (Cv(t))') = 0 \qquad (2.17)$$

(2.17) known as the Lieńard equation.

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Finding the Liénard equation, given these conditions, will lead to the Van der Pol equation.

Setting up the ODE

We again apply Kirchhoff's voltage law, apply voltage drop equations, and apply the derivative w.r.t time to get

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$$F'(I)I' + LI'' + \frac{1}{C}Q' = 0 \tag{3.1}$$

$$\Downarrow$$

$$F'(I)I' + LI'' + \frac{1}{C}I = 0 \tag{3.2}$$

Now let us consider

$$F(I) = \frac{1}{3}I^3 - aI,\ a > 0 \text{ positive constant.} \tag{3.3}$$

Thus, (3.2) becomes:

$$(I^2 - a)I' + LI'' + \frac{1}{C}I = 0 \tag{3.4}$$

Deriving the Van der Pol Equation

We will apply the injective transformation

$$(I, t) \to (\alpha x, \delta s) \tag{3.5}$$

$$\alpha > 0 \text{ and } \delta > 0 \text{ such that } LC = \delta^2 \text{ and } a = \alpha^2$$

Under this transformation, the equation below

$$(I^2 - a)I' + LI'' + \frac{1}{C}I = 0 \tag{3.6}$$

becomes

$$L(\frac{d^2\alpha x}{dt^2}) + ((\alpha x)^2 - \alpha^2)\frac{d(\alpha x)}{dt} + \frac{1}{C}(\alpha x) = 0 \tag{3.7}$$

$$\Downarrow$$

Deriving the Van der Pol Equation cont.

$$\frac{d^2x}{ds^2} + \frac{C\alpha^2}{\delta}((x)^2 - 1)\frac{dx}{ds} + x = 0 \tag{3.8}$$

Letting $\mu = \frac{C\alpha^2}{\delta}$ and $\frac{dx}{ds} = \dot{x}$ in (3.8) we get

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{3.9}$$

which is the Van der Pol equation.

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What is the First Order Averaging Method

- Using the average value of a system over a period to deduce information about the original system
- First order refers to the first derivative

Introduction to First Order Averaging Method

To begin, we will discuss energy conservation in the simple harmonic oscillator as a primer:

$$\ddot{x} + x = 0 \tag{4.1}$$

$$x(0) = \frac{x_0}{2} \tag{4.2}$$

$$\dot{x}(0) = \frac{x_0}{2} \tag{4.3}$$

In order to approach the first order averaging method, we will discuss the energy and average energy of the system.
The total mechanical energy is given by

$$E = E_k + E_p \tag{4.4}$$

where $E_k = \frac{1}{2}\dot{x}^2$ and $E_p = \frac{1}{2}x^2$.

Energy Conservation in the Simple Harmonic Oscillator

Let's see how the total energy changes in time:

$$E = E_k + E_p = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 \tag{4.5}$$

$$\Downarrow$$

$$\frac{dE}{dt} = \dot{x}\ddot{x} + x\dot{x} = \dot{x}(\ddot{x} + x) \tag{4.6}$$

Since we are working with the simple harmonic oscillator, $\ddot{x} + x = 0$. Thus,

$$\frac{dE}{dt} = 0 \tag{4.7}$$

Modifying the Simple Harmonic Oscillator

We introduce a friction coefficient dependent on x to our ODE (4.1):

$$\ddot{x} - \phi\dot{x} + x = 0 \tag{4.8}$$

where $\phi = \mu(1 - x^2)$, $\mu \in \mathbb{R}$, is called the control parameter.
We want to see if the changed system (4.8) is still conservative.

Checking for Conservation of Energy

We start with

$$\ddot{x} - \phi\dot{x} + x = 0 \tag{4.9}$$

and multiply both sides by $\dot{x}$:

$$\ddot{x}\dot{x} - \phi\dot{x}^2 + x\dot{x} = 0 \tag{4.10}$$

$$\ddot{x}\dot{x} + x\dot{x} = \phi\dot{x}^2 \tag{4.11}$$

$$\frac{d(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2)}{dt} = \phi\dot{x}^2 \tag{4.12}$$

where $\frac{dE}{dt} = \frac{d(\frac{1}{2}\dot{x}^2+\frac{1}{2}x^2)}{dt}$. Hence, $\frac{dE}{dt} = \phi\dot{x}^2$, therefore the energy is no longer conserved.

Energy of the Van der Pol Oscillator

Note that the system

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{4.13}$$

is the Van der Pol equation derived in the previous section. Thus the rate of change of the energy of the Van der Pol equation is

$$\frac{dE}{dt} = \mu(1 - x^2)\dot{x}^2 \tag{4.14}$$

We have two distinct cases for energy in (4.14)

- Case 1: $-1 < x < 1$, then $\frac{dE}{dt} > 0$
- Case 2: $x < -1$ or $x > 1$, then $\frac{dE}{dt} < 0$

Energy of the Van der Pol Oscillator cont.

- In case 1, where $\frac{dE}{dt} > 0$, we can see that the amplitudes of the displacement are increasing in time.

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- As well, intuitively we can note that as $x \to \pm 1$, there will be a limit cycle separating the two cases.

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- In case 2, where $\frac{dE}{dt} < 0$, we can see that the amplitudes of the displacement are decreasing in time.
- As well, intuitively we can note that as $x \to \pm 1$, there will be a limit cycle separating the two cases.
- We expect that energy gain and loss over the limit cycle will balance out over one period, thus $\overline{\frac{dE}{dt}} = 0$.

Showing the Average Energy is Zero

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Next, we multiply both sides by $\dot{x}$ to get:

$$\ddot{x}\dot{x} - \mu(1 - x^2)\dot{x}^2 + x\dot{x} = 0$$

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Rearranging, we get:

$$\mu(1 - x^2)\dot{x}^2 = \ddot{x}\dot{x} + x\dot{x}$$

Showing the Average Energy is Zero cont.

Next, to represent averaging over one period (T), we take the integral over $[t_0, t_0 + T]$

$$\int_{t_0}^{t_0+T} \mu(1-x^2)\dot{x}^2 dt = \int_{t_0}^{t_0+T} \ddot{x}\dot{x} + x\dot{x} dt$$

$$\int_{t_0}^{t_0+T} \frac{dE}{dt} dt = \int_{t_0}^{t_0+T} \frac{d(\dot{x}^2 + x^2)}{dt} dt \tag{4.15}$$

We can evaluate the right hand side of the equation:

$$\int_{t_0}^{t_0+T} \frac{d(\dot{x}^2 + x^2)}{dt} dt = \int_{t_0}^{t_0+T} d(\dot{x}^2 + x^2) \qquad = \dot{x}^2 + x^2 \Big|_{t_0}^{t_0+T} \tag{4.16}$$

Since we are working on the limit cycle,

$$\dot{x}^2 + x^2 \Big|_{t_0}^{t_0+T} = 0 \tag{4.17}$$

Showing the Average Energy is Zero cont.

We arrive at

$$\int_{t_0}^{t_0+T} \frac{dE}{dt} dt = 0$$
$$\frac{1}{T} \int_{t_0}^{t_0+T} \frac{dE}{dt} dt = 0$$
$$\frac{1}{T}(E(t_0+T) - E(t_0)) = 0 \tag{4.18}$$

The left hand side of (4.18) is the average value of $\frac{dE}{dt}$ over one period.

Showing the Average Energy is Zero cont.

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The left hand side of (4.18) is the average value of $\frac{dE}{dt}$ over one period. Thus,

$$\overline{\frac{dE}{dt}} = 0 \tag{4.19}$$

Viewing the Van der Pol Equation as perturbations from the Simple Harmonic Oscillator

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$$x = x_h + \mu u(t, x_h) \tag{4.20}$$

where $u \in C^2[t_0, t_0 + T] \times D$, $D \subset \mathbb{R}$
We want to see what happens to the average energy under this new view of x.

Average Energy of the VDPE as perturbations of the SHO

We start with $\overline{\frac{dE}{dt}} = 0$ and sub in $x = x_h + \mu u(t, x_h)$

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We start with $\overline{\frac{dE}{dt}} = 0$ and sub in $x = x_h + \mu u(t, x_h)$

$$\begin{aligned}
\overline{\frac{dE}{dt}} &= 0 \\
\frac{1}{T}\int_{t_0}^{t_0+T} \frac{dE}{dt} dt &= 0 \\
\int_{t_0}^{t_0+T} \mu(1 - x^2)\dot{x}^2 dt &= 0 \\
\int_{t_0}^{t_0+T} \mu(1 - (x_h + \mu u(t, x_h))^2)\frac{(d(x_h + \mu u(t, x_h)))}{dt}^2 dt &= 0 \\
\mu \int_{t_0}^{t_0+T} \dot{x}^2(1 - x_h^2)dt - \mu^2 \int_{t_0}^{t_0+T} \dot{x}^2[2x_h u(t, x_h) - \mu(u(t, x_h))^2)\cdot & \\
(1 + 2\mu u(t, x_h) + (\mu\dot{u}(t, x_h))^2)]dt &= 0
\end{aligned} \tag{4.21}$$

Average Energy of the VDPE as perturbations of the SHO cont.

We get

$$\mu \int_{t_0}^{t_0+T} \dot{x}^2(1 - x_h^2)dt + O(\mu^2) = 0 \tag{4.22}$$

We will use this equation later on to derive the radius of the limit cycle.

Van der Pol equation as an Autonomous System

The Van der Pol equation can be written as an autonomous system of ordinary differential equations as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \mu(1 - x^2)y \end{cases} \tag{4.23}$$

The only equilibrium point of (4.23) is the origin. To show this, let's solve the system

$$\begin{cases} 0 = y \\ 0 = -x + \mu(1 - x^2)y. \end{cases} \tag{4.24}$$

combining both equations, we get

$$0 = -x + \mu(1 - x^2)0 \tag{4.25}$$

$$0 = -x \tag{4.26}$$

Thus, $(0, 0)$ is the only equilibrium point.

Linearizing the Van der Pol Equation

Next, we need to linearize the system about the origin. To do this, we start by finding the Jacobian matrix.

$$\text{Let } f_1(x,y) = y \text{ and } f_2(x,y) = -x + \mu(1-x^2)y$$

Thus the Jacobian matrix of our system is

$$J_{f_1,f_2}(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2\mu xy & \mu(1-x^2) \end{bmatrix}. \tag{4.27}$$

Linearizing the Van der Pol Equation cont.

Next, we need to find the eigenvalues of $J_{f_1,f_2}(0,0)$

Let $A = J_{f_1,f_2}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$

Now we find the eigenvalues of A

$$\begin{aligned} \lambda I - A &= \begin{bmatrix} \lambda & -1 \\ 1 & \lambda - \mu \end{bmatrix} \\ &\Downarrow \\ |\lambda I - A| &= \lambda(\lambda - \mu) + 1 \\ &= \lambda^2 - \lambda\mu + 1 \end{aligned} \tag{4.28}$$

Thus, the eigenvalues of A are

$$\lambda_{1,2} = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - 1} \tag{4.29}$$

Checking Stability of the Origin

We have $\lambda_{1,2} = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - 1}$
To check the stability of the origin, we have to consider 3 different cases:

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- Case 1: $0 < \mu < 2$, then $\left(\frac{\mu}{2}\right)^2 < 1$. This shows that A has imaginary eigenvalues with positive real parts. The origin will be an unstable focus.

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- Case 1: $0 < \mu < 2$, then $\left(\frac{\mu}{2}\right)^2 < 1$. This shows that A has imaginary eigenvalues with positive real parts. The origin will be an unstable focus.
- Case 2: $\mu = 2$, then $\left(\frac{\mu}{2}\right)^2 - 1 = 0$. This shows A has repeated positive eigenvalues. Thus, in this case the origin is an unstable node.
- Case 3: $\mu > 2$, then $\left(\frac{\mu}{2}\right)^2 > 1$, therefore the eigenvalues of A are real. To check stability, We need to check the signs of both eigenvalues. Thus, we have to check if $\frac{\mu}{2} > \sqrt{(\frac{\mu}{2})^2 - 1}$

$$\frac{\mu}{2} > \sqrt{(\frac{\mu}{2})^2 - 1} \Leftrightarrow \left(\frac{\mu}{2}\right)^2 > \left(\frac{\mu}{2}\right)^2 - 1 \Leftrightarrow 0 > -1 \tag{4.30}$$

Thus, both eigenvalues are positive, and the origin is an unstable node.

Checking Stability of the Origin cont.

In each of the three cases, the origin is unstable. We can conclude that when $\mu > 0$, we have an unstable equilibrium point at the origin.

Finding the Radius of the Limit Cycle

In the limit cycle we view the oscillatory process as energy conservative, therefore we can claim the following:

$$x_h^2 + \dot{x}_h^2 = E_b \tag{4.31}$$

For simplicity we can rewrite (4.31) as

$$x_h^2 + \dot{x}_h^2 = \phi_0^2, \text{ where } \phi_0 = \sqrt{E_b} \neq 0 \tag{4.32}$$

Returning to the equation $\mu \int_{t_0}^{t_0+T} (1 - x^2)\dot{x_h}^2 dt + O(\mu^2) = 0$, and noting that $0 < \mu << 1$, so $O(\mu^2)$ can be approximated to zero, we get

$$\int_{t_0}^{t_0+T} (1 - x_h^2)\dot{x}_h^2 dt = 0 \tag{4.33}$$

Finding the Radius of the Limit Cycle cont.

We want to find the radius of the circular orbit, ϕ_0, such that the integral $\int_{t_0}^{t_0+T}(1-x_h^2)\dot{x}_h^2 dt$ is zero, independently of the period $T = 2\pi$ of the limit cycle.

Finding the Radius of the Limit Cycle cont.

We want to find the radius of the circular orbit, ϕ_0, such that the integral $\int_{t_0}^{t_0+T}(1-x_h^2)\dot{x}_h^2 dt$ is zero, independently of the period $T = 2\pi$ of the limit cycle.
Since we are working with the equation $x_h^2 + \dot{x}_h^2 = \phi_0^2$, we can parameterize this orbit as follows:

$$\begin{cases} x_h = \phi_0 \cos(\theta) \\ \dot{x}_h = \phi_0 \sin(\theta) \end{cases} \tag{4.34}$$

Finding the Radius of the Limit Cycle cont.

Using the parameterization, we can find $\frac{d\theta}{dt}$

$$x_h = \phi_0 \cos(\theta)$$

$$\dot{x}_h = -\phi_0 \sin(\theta) \frac{d\theta}{dt}$$

and

$$\dot{x}_h = \phi_0 \sin(\theta)$$

Therefore, we get

$$\frac{d\theta}{dt} = -1. \tag{4.35}$$

and

$$\theta(t) = -t + c,\ c \in \mathbb{R} \tag{4.36}$$

Finding the Radius of the Limit Cycle cont.

We have enough information to solve the integral to find the radius.

$$\begin{aligned}\int_{t_0}^{t_0+T}(1-x_h^2)\dot{x}_h^2 dt &= \int_{t_0}^{t_0+T}(1-\phi_0^2\cos^2(\theta))\phi_0^2\sin^2(\theta)dt \\ &= \phi_0^2\int_{t=t_0}^{t=t_0+T}(\sin^2(\theta)-\phi_0^2\sin^2(\theta)\cos^2(\theta))\frac{dt}{d\theta}d\theta \\ &= \phi_0^2\left(\frac{4-\phi_0^2}{8}\right)T \end{aligned} \tag{4.37}$$

In order for (4.37) to equal zero, we need $\phi_0 = 2$. Hence, for $0 < \mu << 1$, the limit cycle is a circle of radius 2.

Plots of Van der Pol equation I

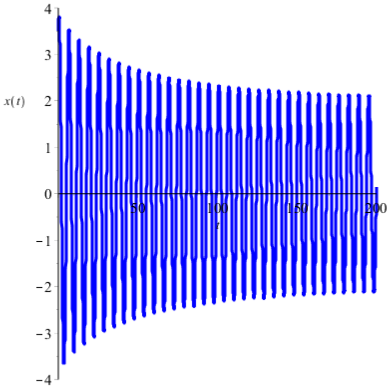

Figure: Energy dissipation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 3.5$ and $\dot{x}(0) = 1.5$.

Plots of Van der Pol equation II

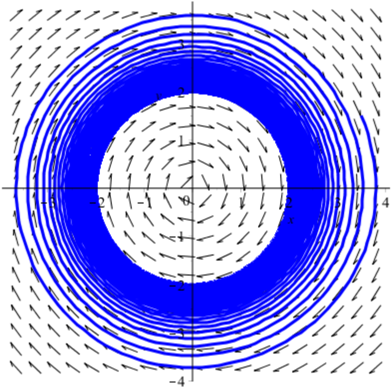

Figure: Phase portrait for energy dissipation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 3.5$ and $\dot{x}(0) = 1.5$.

Plots of Van der Pol equation III

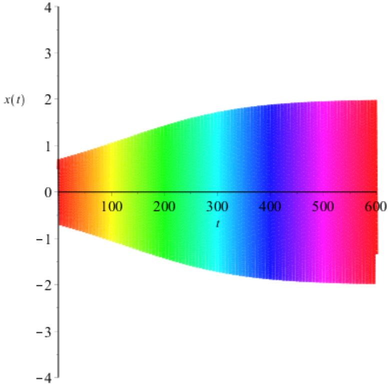

Figure: Energy generation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 0.5$ and $\dot{x}(0) = 0.5$.

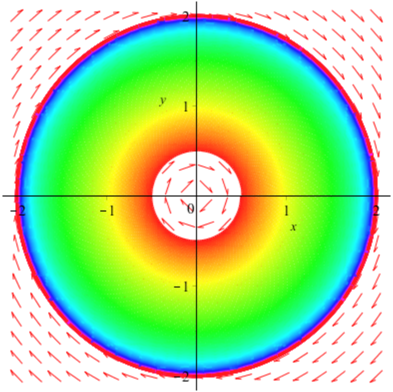

Figure: Phase portrait for energy generation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 0.5$ and $\dot{x}(0) = 0$.

Conclusion

- Solved a classical dynamical systems/ODEs problem using an original approach

Conclusion

- Solved a classical dynamical systems/ODEs problem using an original approach
- Gained an appreciation for perturbation theory, which allowed us to continue onto the analytical approach.

Thanks!

Thank you for your time!

References I

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