The Variation of Pressure with Respect to Altitude

Ana Mucalica, MacEwan University

Faculty mentor: Dr. Ion Bica

Work done in collaboration with Anneliese Ansorger, Cory Efird, Cassandra Lisitza, Christopher Macyk, Tarig Mergani, Adam Reeves, Rebecca Walton, Yaying Zhong, and Jett Ziehe, MacEwan University

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Static model describing the variation of pressure with respect to altitude in the atmosphere, when the fluid is in hydrostatic balance

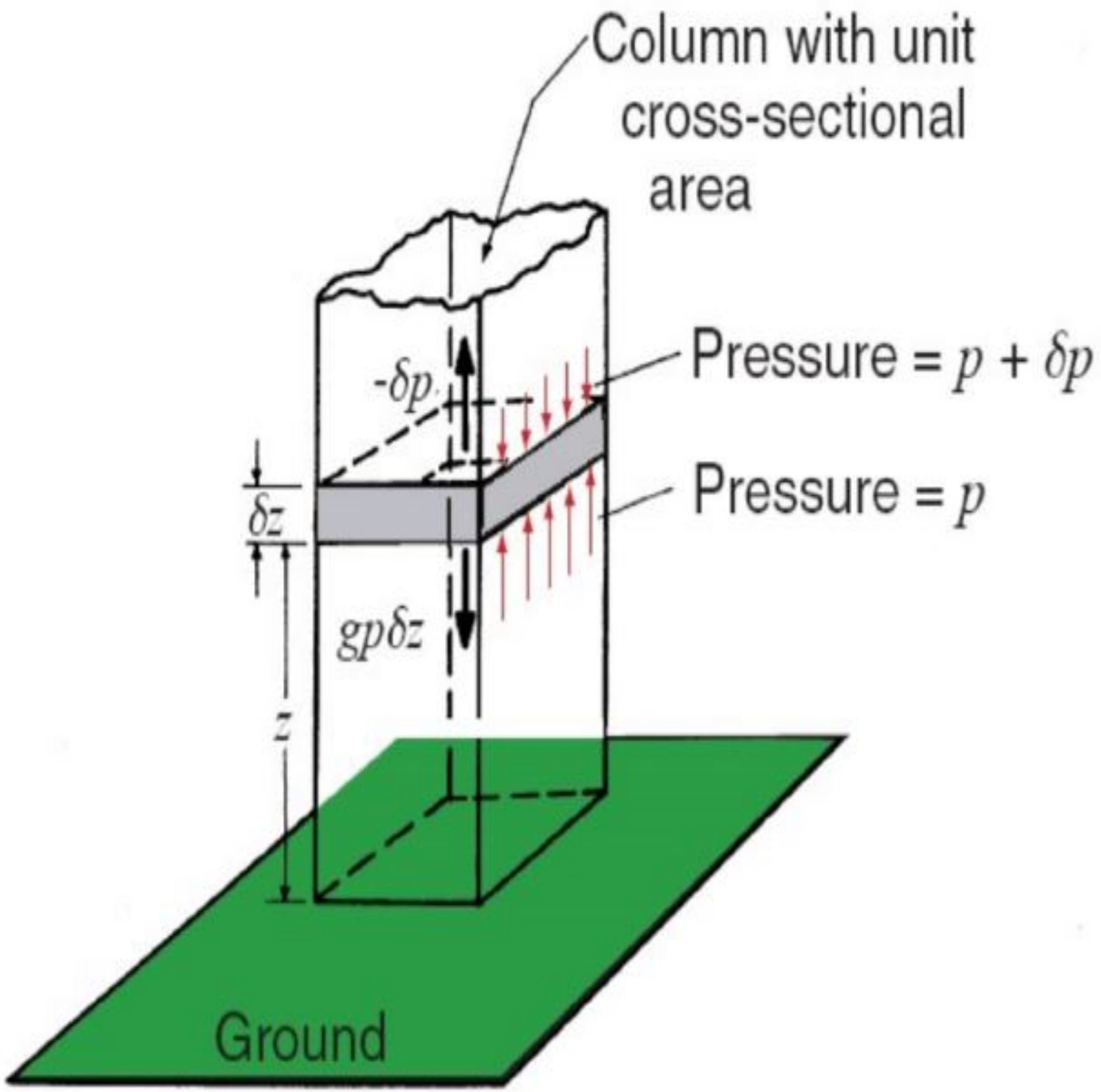
$$p\frac{\partial f(h,p)}{\partial p}-\frac{R\cdot T(h)}{M\cdot g}\frac{\partial f(h,p)}{\partial h}=0$$

Hydrostatic Balance

The hydrostatic balance occurs when the pressure at any point in the fluid equals the weight of an air column of unit section from above the point.

Newton's second law of motion

$$\frac{d\vec{v}}{dt}=\frac{\sum\vec{F}}{m}$$



Consider a controlled volume of air (air column) satisfying the following properties:

1. It has a constant mass density ρ.

2. It has a constant cross-sectional area A.

3. The turbulence in the column is negligible.

4. The column has height H.

Due to the molecular movements across the surfaces of the control volume of air (dV=dxdydz), pressure (p) is exerted in all three directions, i.e.

$$p = p(x, y, z).$$

Let $\vec{F}(x, y, z)$ be the pressure gradient force that acts on our column of air, then we have:

$$\frac{\vec{F}}{dV} = -\nabla p = -\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Looking at the vertical dependence of the pressure field in the real atmosphere, we simplify the equation of motion of the air by disregarding the effects of the Coriolis force, centrifugal force and friction forces, therefore we have:

$$\frac{\vec{F}}{dV} = -\begin{bmatrix} 0 \\ 0 \\ p_z \end{bmatrix}$$

With further manipulations and using the fact that the mass of the control volume of air is m = ρdV, we have

$$\overrightarrow{F_p} = -\begin{bmatrix} 0 \\ 0 \\ \frac{m}{\rho} \cdot \frac{\partial p}{\partial z} \end{bmatrix}$$

Applying Newton's second law, we get:

$$\frac{dv}{dt} = \frac{\overrightarrow{F_p} + \overrightarrow{F_g}}{m}$$

$$\overrightarrow{F_g} = -\begin{bmatrix} 0 \\ 0 \\ g \cdot m \end{bmatrix}$$

Again, performing standard manipulations we arrive at

$$-\rho \frac{dv}{dt} = \begin{bmatrix} 0 \\ 0 \\ \rho \cdot g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\partial p}{\partial z} \end{bmatrix}$$

The vertical velocity at synoptic scale can not be directly measured, but its magnitude can be evaluated from the horizontal velocity, which was disregarded, thus we have $\frac{dv}{dt} = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho \cdot g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\partial p}{\partial z} \end{bmatrix} \Rightarrow 0 = \rho \cdot g + \frac{\partial p}{\partial z}$$

$$\frac{\partial p}{\partial z} = -\rho \cdot g$$

Solving the PDE

First order linear and homogeneous PDE in standard form:

$$A_1(\vec{x})\frac{\partial u}{\partial x_1} + A_2(\vec{x})\frac{\partial u}{\partial x_2} + \cdots + A_n(\vec{x})\frac{\partial u}{\partial x_n} = 0$$

We work with the PDE under the following conditions:

1. $A_i \in C^1(D),\ i = 1 \dots n$
2. There is no $\vec{x} \in \mathrm{D}$ such that $A_i(\vec{x}) = 0,\ i = 1 \dots n$, i.e. we have the following inequality:

$$\sum_{i=1}^{n} A_i^2(\vec{x}) > 0 \ \ \forall \vec{x} \in D$$

So for our model to be valid from a mathematical point of view we need

$$p^2 + \left(\frac{R \cdot T(h)}{M \cdot g}\right)^2 > 0$$

to be satisfied for any region of the atmosphere. This inequality holds because p can never be zero, i.e., for any region of the atmosphere, the pressure must exist.

Now we look for a general solution of the PDE. We have the following characteristic system:

$$\frac{dp}{p} = -\frac{dh}{\left(\frac{R \cdot T(h)}{M \cdot g}\right)}$$

$$\Rightarrow \frac{dp}{dh} + p\,\frac{M \cdot g}{R \cdot T(h)} = 0$$

Then with the choice of integrating factor, $\mu(h) = e^{\int \frac{M \cdot g}{R \cdot T(h)} dh}$, we have the following:

$$y(h) = \int \frac{M \cdot g}{R \cdot T(h)} dh + K, \;\; K \in \mathrm{R}$$

Now we apply the initial conditions $y(h_0) = y_0$, so that our integrating factor becomes

$$\mu(h) = e^{\int_{h_0}^{h} \frac{M \cdot g}{R \cdot T(\xi)} d\xi + y_0}$$

Multiplying the ODE by this integrating factor we obtain the characteristic curve of the PDE

$$\psi(h, p) = p \cdot e^{\int_{h_0}^{h} \frac{M \cdot g}{R \cdot T(\xi)} d\xi + y_0}$$

Hence the general solution of the PDE is

$$f(h,p) = \phi\left(p \cdot e^{\int_{h_0}^{h} \frac{M\cdot g}{R\cdot T(\xi)} d\xi + y_0}\right),\ \phi \in C^1(\Omega),\ \Omega \subset \mathbb{R}$$

where ϕ is an arbitrary function.

Consider the Cauchy problem for the PDE, with the initial condition

$$f(h_0, p) = p = u(p)$$

By setting h=h_0 in our characteristic curve and solving for

$p = \omega(C)$, we get

$$p = C \cdot e^{-y_0} = \omega(C)$$

Hence, the solution to the Cauchy problem is

$$u = \phi(\omega(\psi)) = \omega(\psi) = \psi \cdot e^{-y_0} = p \cdot e^{\int_{h_0}^{h} \frac{M \cdot g}{R \cdot T(\xi)} d\xi} \cdot e^{-y_0}$$

$$\Rightarrow u = p \cdot e^{\int_{h_0}^{h} \frac{M \cdot g}{R \cdot T(\xi)} d\xi} = F(h, p)$$

Now we solve the equation $F(h, p) = p_0$ for p.

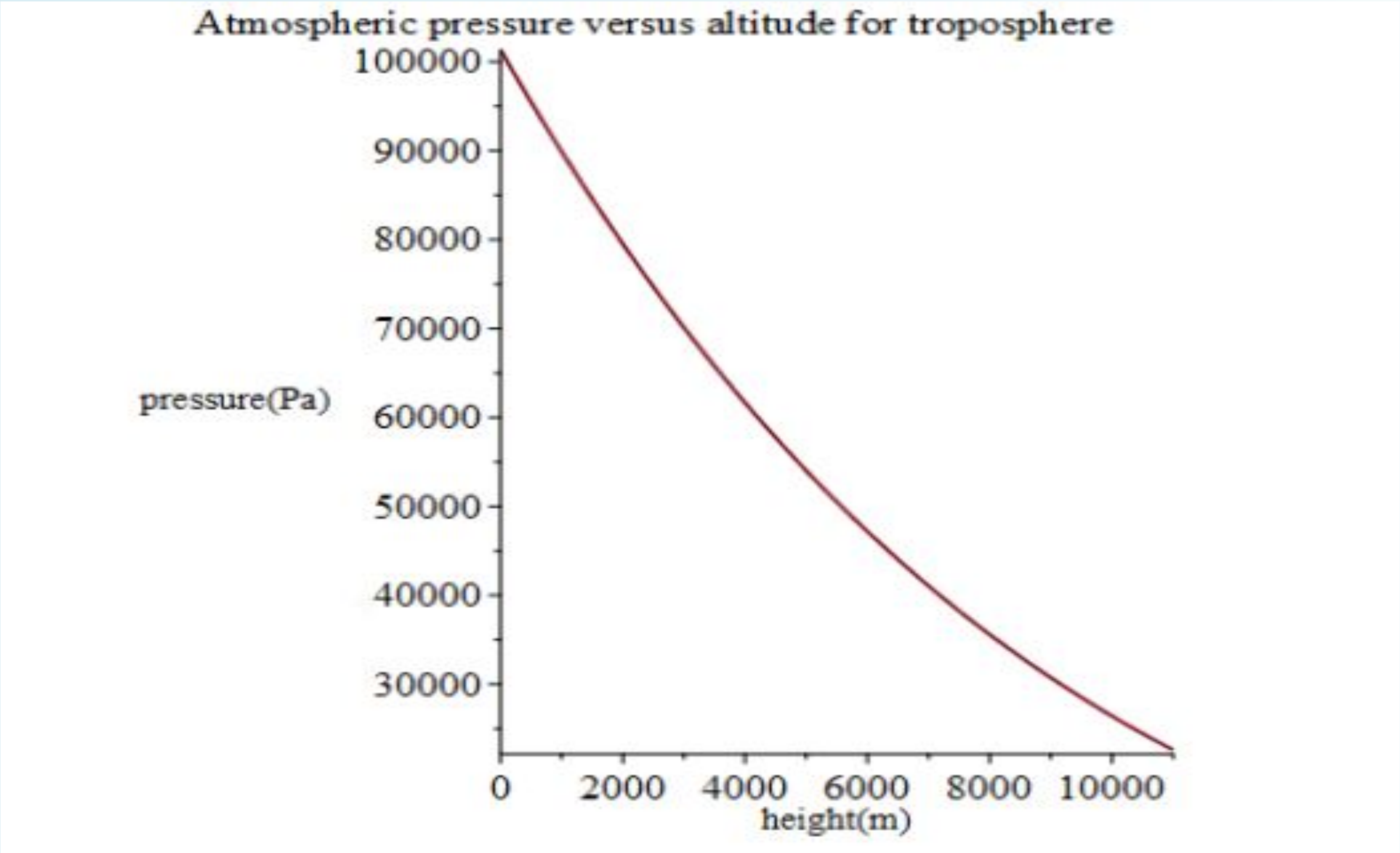
$$p(h) = p_0 \cdot e^{-\int_{h_0}^{h} \frac{M \cdot g}{R \cdot T(\xi)} d\xi}$$

The latter equation represents the solution of the hydrostatic equation with the initial condition

$$p(h_0) = p_0.$$

Troposphere

The troposphere is the lowest atmospheric blanket, which is connected to the Earth's surface. Using our mathematical model we give the pressure estimates for the troposphere (0m - 11000m) below.

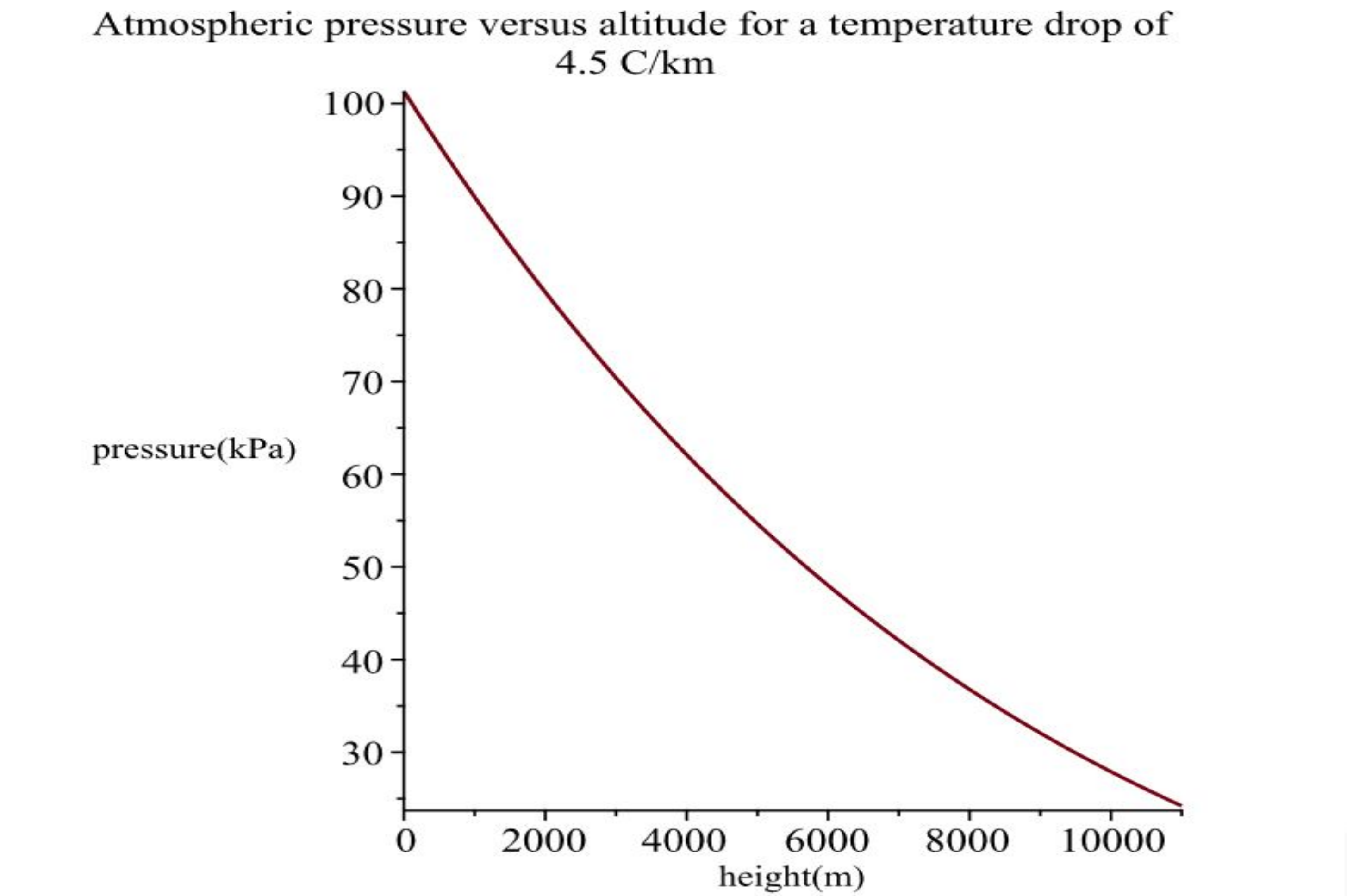


We observe the following trend. As altitude increases, atmospheric pressure decreases. Thus, these two variables are inversely proportional.

Altitude Above Sea Level			Temperature		Atmospheric Pressure
Feet	Miles	Meters	F	C	kPa
0		0	59	15	101.32
500		153	57	14	99.50
1000		305	55	13	97.71
1500		458	54	12	95.94
2000		610	52	11	94.21
2500		763	50	10	92.49
3000		915	48	9	90.80
3500		1068	47	8	89.14
4000		1220	45	7	87.50
4500		1373	43	6	85.88
5000	0.95	1526	41	5	84.29
6000	1.1	1831	38	3	81.18
7000	1.3	2136	34	1	78.16
8000	1.5	2441	31	-1	75.24
9000	1.7	2746	27	-3	72.40
10,000	1.9	3050	23	-5	69.66
15,000	2.8	4577	6	-14	57.14
20,000	3.8	6102	-12	-24	46.52
25,000	4.7	7628	-30	-34	37.56
30,000	5.7	9153	-48	-44	30.05
35,000	6.6	10,679	-66	-54	23.80
36089	6.8	11,000	-70	-56	22.63

Temperature Change Scenario

Temperature in the troposphere drops an average of 6.5°C per kilometer of altitude, however in this section we hypothesized a climate change scenario where the temperature drops an average of 4.5°C per kilometer of altitude. Using our mathematical model we obtain higher atmospheric pressures.



When comparing these values with the standard atmospheric pressures, we can see that when the temperature is increased by 2.0°C, the pressure increases as well. More precisely, it increases by an average of 3.047 Pa.

Altitude Above Sea Level			Temperature		Atmospheric Pressure
Feet	Miles	Meters	F	C	kPa
0		0	59	15	101.325
500		153	57	14	99.501328
1000		305	55	13	97.717865
1500		458	54	12	95.950766
2000		610	52	11	94.222757
2500		763	50	10	92.510733
3000		915	48	9	90.836709
3500		1068	47	8	89.178298
4000		1220	45	7	87.556822
4500		1373	43	6	85.950591
5000	0.95	1526	41	5	84.370008
6000	1.1	1831	38	3	81.294316
7000	1.3	2136	34	1	78.316451
8000	1.5	2441	31	-1	75.643651
9000	1.7	2746	27	-3	72.643651
10,000	1.9	3050	23	-5	69.952297
15,000	2.8	4577	6	-14	57.701841
20,000	3.8	6102	-12	-24	47.371336
25,000	4.7	7628	-30	-34	38.680981
30,000	5.7	9153	-48	-44	31.414142
35,000	6.6	10,679	-66	-54	25.359704
36089	6.8	11,000	-70	-56	24.224061

Mathematical atmospheric models for predicting atmospheric properties are essential to trades like the aviation industry, as even small increases in pressures can have a significant impact on the calibration of aircrafts' instrumentation systems. International Standard Atmosphere (ISA) is a model used for the standardization of aircraft instruments. Flying in ISA-plus temperatures has a negative impact on aircraft performance because if ISA-plus temperatures are excessive, aircraft may not climb at the anticipated rate and/or may be unable to maintain altitude.

Change in atmospheric pressure can also have an impact on the human body. More specifically, on blood pressure. This is because low temperatures cause our blood vessels to narrow, which in turn increases blood pressure. So, at higher temperatures, with higher atmospheric pressure, we could expect a decrease in blood pressure. This can have negative effects on people suffering from conditions such as hypotension. It is worth noting that even though these climate changes would directly impact present generations, the future generations would likely experience biological adaptation, meaning the organisms would gradually alter bodily functions to acclimatize to such changes in atmosphere.

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THANK YOU!